MODULE-4

- Feynman Rules for Quantum Electrodynamics (QED)
- Elastic electron-proton scattering
- Inelastic electron-proton scattering

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Quantum Electrodynamics

Feynman Rules for QED (Interaction between fermions and photons)

To compute the amplitude , M, associated with a particular Feynman diagram (FD), we proceed as follows:

$$\Psi \sim b(k,s)u(k,s)e^{-ikx} + d^{\dagger}(k,s)v(\mathrm{k,s})\mathrm{e}^{\mathrm{ikx}}$$

$b, b^{\dagger}$ represent operators for annihilation of particle and creation of particle

$d, d^{\dagger}$ represent operators for annihilation of antiparticle and creation of antiparticle

$$\mathcal{L}_{QED} = i\bar{\Psi}[ig\gamma^{\mu}]\Psi A_{\mu}$$

1. *Notation*. Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$, and the corresponding spins $s_1, s_2, \ldots, s_n$; label the internal four-momenta $q_1, q_2, \ldots$. Assign arrows to the lines as follows: the arrows on *external* fermion lines indicate whether it is an electron or a positron; arrows on *internal* fermion lines are assigned so that the "direction of the flow" through the diagram is preserved (i.e., every vertex must have one arrow entering and one arrow leaving). The arrows on external

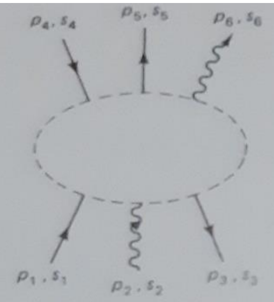


Figure 7.1 A typical QED diagram, with external lines labeled. (Internal lines not shown.)

photon lines point "forward"; for internal photon lines the choice is arbitrary. (See Fig. 7.1.)

2. *External Lines*. External lines contribute factors as follows:

Electrons: Incoming ( ): u; Outgoing ( ): $\bar{u}$

Positrons: Incoming ( ): $\bar{v}$; Outgoing ( ): v

Photons: Incoming ( ): ϵ^{μ}; Outgoing ( ): $\epsilon^{\mu *}$

3. *Vertex Factors.* Each vertex contributes a factor

$$ig_e\gamma^\mu$$

The dimensionless coupling constant g_e is related to the charge of the positron: $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$.*

4. *Propagators.* Each internal line contributes a factor as follows:

Electrons and positrons: $$\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$

Photons: $$\frac{-ig_{\mu\nu}}{q^2}$$

5. *Conservation of Energy and Momentum.* For each vertex, write a delta function of the form

$$(2\pi)^4\,\delta^4(k_1 + k_2 + k_3)$$

where the k's are the three four-momenta coming *into* the vertex (if an arrow leads *outward*, then k is *minus* the four-momentum of that line, except for external positrons*). This factor enforces conservation of energy and momentum at the vertex.

6. *Integrate Over Internal Momenta.* For each internal momentum q, write a factor

$$\frac{d^4q}{(2\pi)^4}$$

and integrate.

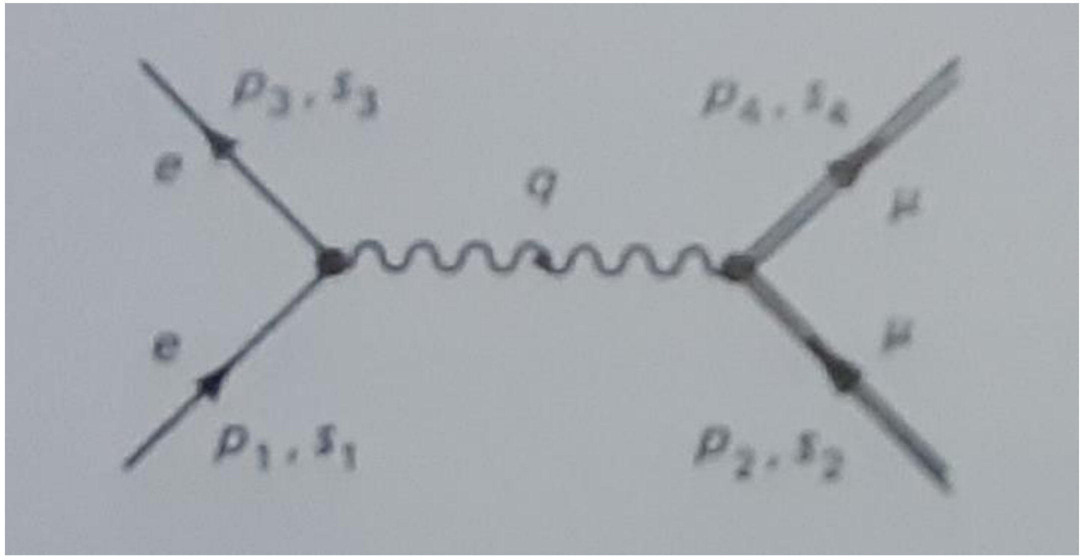
7. *Cancel the Delta Function.* The result will include a factor

$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$

corresponding to overall energy-momentum conservation. Cancel this factor, and what remains is $-i\mathcal{M}$.

8. *Antisymmetrization.* Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with an outgoing positron (or vice versa).

$e - \mu$ Scattering Process



$$(2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\nu)u^{(s_2)}(p_2)] \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4)d^4q$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$

Continued…

$\langle |M|^2 \rangle$ = Average of initial spin & sum over final states

- Initial spins being random, average over initial spins are taken
- In final states particles are detected in a particular direction, hence those are specified. Sum of final spins are considered.

$$|M|^2 = \frac{g_e^4}{(p_1 - p_3)^4}[\bar{u}(3)\gamma^\mu u(1)]\big[\bar{u}(4)\gamma_\mu u(2)\big][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^*$$

Let us simplify the product of either 1st and 3rd square bracketed term or 2nd and 4th ones.

Consider a general expression,

$G = [\bar{u}\,(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$; a & b stand for spins and momenta Γ_1 and Γ_2 are 4×4 matrices.

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [\bar{u}(a)\Gamma_2 u(b)]^\dagger = \big[u^\dagger(a)\gamma^0\Gamma_2 u(b)\big]^\dagger$$

$$= u^\dagger(b)\Gamma_2^\dagger(\gamma^0)^\dagger u(a) = u^\dagger(b)\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a)$$

Using $(\gamma^0)^2 = 1$ & $(\gamma^0)^\dagger = \gamma^0$ and defining $\bar{\Gamma}_2 = \gamma^0\,\Gamma_2^\dagger\gamma^0$

$\therefore G = [\bar{u}\,(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = [\bar{u}\,(a)\Gamma_1 u(b)][\bar{u}(b)\,\bar{\Gamma}_2 u(a)]$

$\sum_s u\bar{u} = (\not{p} + m)$; $\sum_s v\bar{v} = (\not{p} - m)$ and $\not{p} = p^\mu\gamma_\mu$

Continued…

$\sum_{s_b} G = \bar{u}(a)\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2 u(a) = \bar{u}(a)Qu(a)$ defining $Q = \Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2$

$$\sum_{s_a}\sum_{s_b} G = \sum_{s_a} \bar{u}(a)Qu(a) = \sum_{s_a} \bar{u}_i(a)Q_{ij}u_j(a) = Q_{ij}\sum_{s_a} u_j(a)\bar{u}_i(a) = Q_{ij}(\not{p}_a + m_a)_{ji}$$

$$= Tr[Q(\not{p}_a + m_a)] = Tr[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a + m_a)]$$

Now,

$$\langle|M|^2\rangle = \frac{1}{4}\frac{g_e^4}{(p_1-p_3)^4} Tr[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)] \times Tr\left[\gamma_\mu(\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M)\right] \text{ ----------- } (1)$$

Spin average over initial spins , for spin ½ particles . $\frac{1}{(2s_1+1)(2s_2+1)}$

m, M are mass of electron and muon respectively.

This is commonly known as Casimir's Trick of simplification. Now Trace technology is needed to further simplify the above expression.

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1. $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$
2. $g_{\mu\nu}g^{\mu\nu} = 4$
3. $\gamma_{\mu}\gamma^{\mu} = 4$

Trace theorems/formulae needed for simplifications:

1. $Tr(AB) = Tr(BA)$
2. $Tr(ABC) = Tr(BCA) = Tr(CAB)$
3. $Tr(odd\ number\ of\ \gamma\ matrices) = 0$
4. $Tr(1) = 4$
5. $Tr(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}$
6. $Tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

Continued…

From Eq. (1) :

$$Tr[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)] = Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\ ) + m[Tr(\gamma^\mu \not{p}_1 \gamma^\nu\ ) + Tr(\gamma^\mu \gamma^\nu \not{p}_3)] + m^2 Tr(\gamma^\mu \gamma^\nu)$$

- Square bracketed term in the above expression vanishes due to Trace theorem 3.

$$Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\ ) = (p_1)_\lambda (p_3)_\sigma Tr\left(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma\right) = (p_1)_\lambda (p_3)_\sigma 4\left(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu}\right)$$

$$= 4\left(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu\ \right) \text{ using Trace theorem 6}$$

$$\therefore Tr[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)] = 4\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1 \cdot p_3)\right] \text{ using Trace theorem 5}$$

Similarly,

$$Tr\left[\gamma_\mu(\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M)\right] = 4\left[p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + g^{\mu\nu}(M^2 - p_2 \cdot p_4)\right]$$

Now, Eq. (1) becomes,

$$\langle |M|^2 \rangle = \frac{1}{4} \frac{g_e^4}{(p_1 - p_3)^4} 16\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1 \cdot p_3)\right] \times \left[p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + g^{\mu\nu}(M^2 - p_2 \cdot p_4)\right]$$

$$= \frac{8 g_e^4}{(p_1 - p_3)^4}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - M^2(p_1 \cdot p_3) - m^2(p_2 \cdot p_4) + 2m^2 M^2] \text{ --------- } (2)$$

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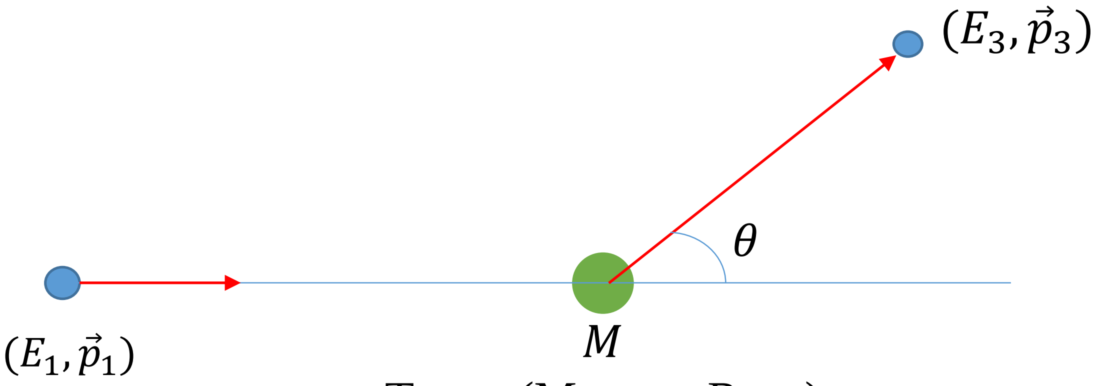
In Laboratory frame, assuming muon as target particle with infinitely heavy and electron as bombarding particle



Target particle does not recoil and assume $E_1 = E_3 = E$

Momentum assignment: $p_1 \equiv (E, \vec{p}_1)$; $p_2 \equiv (M, 0)$; $p_3 = (E, \vec{p}_3)$; $p_4 \equiv (M, 0)$

3-momentum conservation gives : $\vec{p}_1 + 0 = \vec{p}_3 + 0 \Rightarrow |\vec{p}_1| = |\vec{p}_3| = p$ (Let)

$$(p_1 - p_3)^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2 = [|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3| \cos\theta] = -2p^2(1 - \cos\theta) = -4p^2 \sin^2\frac{\theta}{2}$$

$$\therefore (p_1 - p_3)^4 = 16p^4 \sin^4\frac{\theta}{2}$$

Continued…

$p_1 \cdot p_2 = EM$; $p_3 \cdot p_4 = EM$; $p_1 \cdot p_4 = EM$; $p_2 \cdot p_3 = EM$

$$p_1 \cdot p_3 = E^2 - \vec{p}_1 \cdot \vec{p}_3 = E^2 - p^2 \cos\theta = p^2 + m^2 - p^2 \cos\theta = m^2 + 2p^2 \sin^2\frac{\theta}{2}$$

$p_2 \cdot p_4 = M^2$

Now substituting all these in Eq. (2) we get,

$$\langle |M|^2 \rangle = \frac{g_e^4 M^2}{p^4 \sin^4\frac{\theta}{2}}\left[E^2 - p^2 \sin^2\frac{\theta}{2}\right] = \frac{g_e^4 M^2 E^2}{p^4 \sin^4\frac{\theta}{2}}\left[1 - v^2 \sin^2\frac{\theta}{2}\right] \text{ using } v = \frac{p}{E}$$

Plugging the above expression in differential scattering cross-section formula ,

$$\left(\frac{d\sigma}{d\Omega}\right)_{point} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{1}{8\pi M}\right)^2 \langle |M|^2 \rangle = \frac{\alpha^2 E^2}{4p^4 \sin^4\frac{\theta}{2}}\left[1 - v^2 \sin^2\frac{\theta}{2}\right] \text{ ------------ } (3)$$

Where $g_e = \sqrt{4\pi\alpha}$

In the non-relativistic limit, the result matches with the Rutherford formula,

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{\alpha^2 E^2}{4p^4 \sin^4\frac{\theta}{2}} \approx \frac{\alpha^2}{16E^2 \sin^4\frac{\theta}{2}} \qquad \text{assuming } Z = 1$$

Continued…

Eq. (3) shows that in the non-relativistic limit ($v \ll c$) , the result does not depend on electron spin because it agrees with the Rutherford result. It means that for $v \to 0,$ the effect of spin can not be stated. Why ???

The spin direction does not change in the scattering of non-relativistic electrons. This is due to the fact that electrons interact dominantly via electric field which can not flip the spin direction. At higher energies it is the magnetic field which flips the spins. Hence, only in the relativistic limit when v is large enough to be compared with c , the spin flipping occurs due to magnetic field.

Electrons are used as good probes for revealing the substructure of protons. This is because of the fact that electrons do not take part in strong interactions. By measuring the angular distribution of scattered electrons from protons and comparing it with the cross-section for scattering electrons from a point charge we can estimate the charge radius of protons.

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} |F(q)|^2$$; where $q = p_i - p_f$ i.e., momentum transfer between the incident e and p

- $F(q)$ is the Fourier transform of the charge distribution. $\mathrm{F}(\vec{q}) = \int \rho(\vec{x}) e^{i\vec{q}\cdot\vec{x}}\, d^3x$
- Normalization condition : $\int \rho(\vec{x})\, d^3x = 1$
- $F(0) = 1$ using the normalization condition

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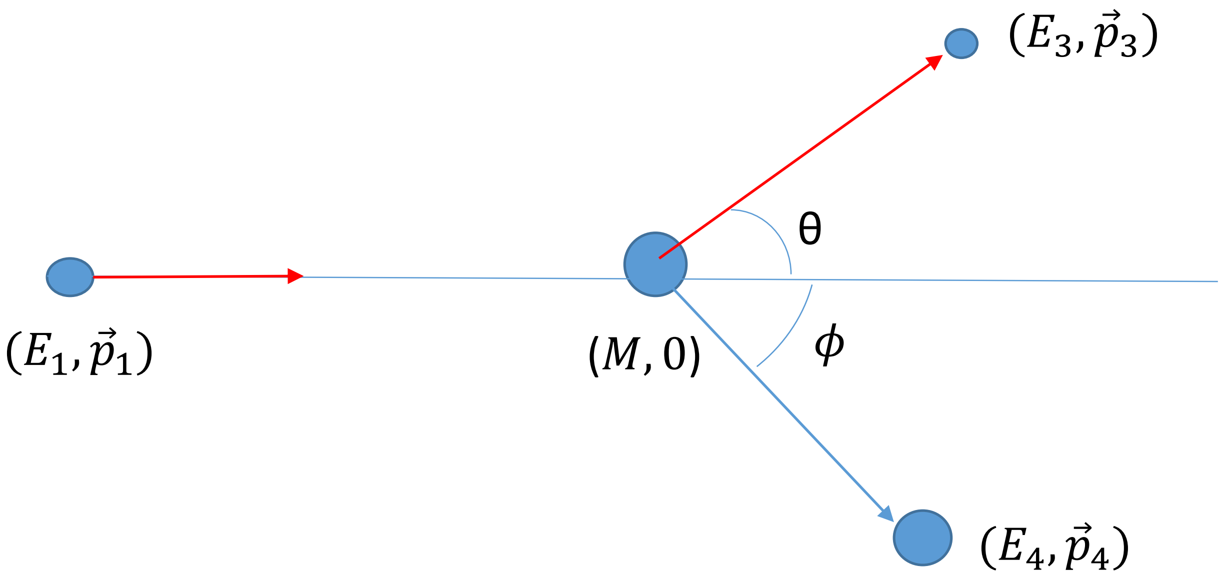


Considering the recoil of muon the pictorial representation of $e - \mu$ scattering in laboratory frame .

Assume $E_1 = E$ & $E_3 = E'$

Neglecting the mass of electron in Eq. (2) we get,

$$\langle |M|^2 \rangle = \frac{8g_e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - M^2(p_1 \cdot p_3)\right] \text{ ---------- } (4)$$

$$E_1 = E \approx |\vec{p}_1| \;; E_3 = E' \approx |\vec{p}_3| \;; p_1^2 = p_3^2 \approx 0 \;; \quad q = p_1 - p_3 = p_4 - p_2$$

$$q^2 = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2EE'(1 - \cos\theta) = -4EE' \sin^2\frac{\theta}{2}$$

Continued…

$$(p_1 \cdot p_2) = EM\ ;\ (p_3 \cdot p_4) = p_3 \cdot (p_1 - p_3 + p_2) = p_3 \cdot p_1 - 0 + p_3 \cdot p_2 = -\frac{q^2}{2} + E'M$$

$$(p_2 \cdot p_3) = E'M\ ;\ (p_1 \cdot p_4) = p_1 \cdot (p_1 - p_3 + p_2) = 0 - p_1 \cdot p_3 + p_1 \cdot p_2 = +\frac{q^2}{2} + EM$$

$$\langle |M|^2 \rangle = \frac{8g_e^4}{q^4}\left[EM\left(-\frac{q^2}{2} + E'M\right) + E'M\left(\frac{q^2}{2} + EM\right) + \frac{M^2 q^2}{2}\right]$$

$$= \frac{8g_e^4}{q^4}\left[2EE'M^2 - \frac{M(E-E')q^2}{2} + \frac{M^2q^2}{2}\right]$$

$$= \frac{8g_e^4}{q^4} 2M^2EE'\left[1 + \frac{q^2}{4EE'} - \frac{q^2}{2M^2}\frac{(E-E')}{2EE'}\right]$$

$$= \frac{8g_e^4}{q^4} 2M^2EE'\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

Continued…

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{1}{8\pi M}\right)^2 \langle |M|^2 \rangle = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right) \frac{E'}{E}\left[\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \text{---------} \quad (5)$$

Problem: Show that $\frac{E'}{E} = \frac{1}{1+\frac{2E}{M}\sin^2\frac{\theta}{2}}$

The above formula is very powerful for exploring the internal structure of a target in which the target is bombarded with a beam of high-energy electrons and to observe the angular distribution and energy of the scattered electrons. Such experiments have enormous impact on understanding the structure of matter. This technique will be used in revealing the substructure of PROTON.

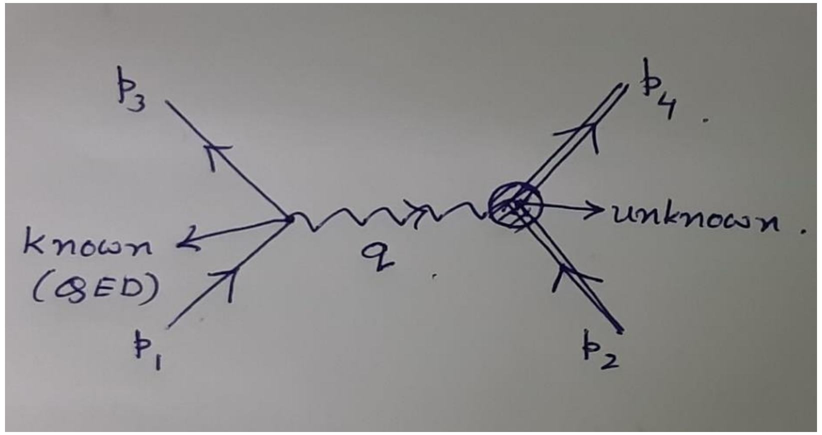
Electron-Proton scattering experiment is very useful in revealing the substructure of proton depending upon the energy of the bombarding electron.

According to de'Broglie hypothesis $\lambda = \frac{h}{\sqrt{2mE}}$ where m is mass of e and E is the energy of the bombarding e.

For probing smaller dimension one requires to increase the energy of the bombarding e.

- $\lambda > d_p$: Electron sees proton as a point particle
- $\lambda \approx d_p$: Electron sees the proton as an extended object instead of a point particle.
- $\lambda < d_p$: Electron sees that proton is made up of more fundamental particles.
- $\lambda \ll d_p$: Electron sees gluons and quarks inside the proton.

Elastic $e - p$ Scattering



If proton were a point like particle , the differential scattering cross-section would have been the same as electron-muon scattering except the fact that muon mass would be replaced by proton mass. Hence,

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{1}{8\pi M}\right)^2 \langle |M|^2 \rangle = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right) \frac{E'}{E}\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

The piece of information unknown to us is how does proton couple with virtual photon. The vertex factor at the proton-photon vertex is denoted by $i\Gamma^\nu$.

Properties that Γ^ν must satisfy :

- Lorentz vector
- Hermiticity
- Gauge invariance /Ward identity

Continued…

QED Vertex factor at electron-photon vertex (Known) : $ig_e\gamma^\mu$

QED Vertex factor at proton-photon vertex (Unknown!!) : $ig_e\Gamma^\mu$

$$\Gamma^\mu = A_1(q^2)\gamma^\mu + A_2(q^2)p_2^\mu + A_3(q^2)p_4^\mu + iA_4(q^2)\sigma^{\mu\nu}p_{2\nu} + iA_5(q^2)\sigma^{\mu\nu}p_{4\nu} \text{ -------- } (6)$$

The coefficients are only function of q^2, other scalars constructed at that vertex may be expressed as function of q^2 and M^2. Note that in Eq. (6) γ_5 term is not considered as parity conservation is enforced.

Consider the term,

$$J^\mu = \bar{u}(4)\Gamma^\mu u(2) = \bar{u}(4)\left[A_1(q^2)\gamma^\mu + A_2(q^2)p_2^\mu + A_3(q^2)p_4^\mu + iA_4(q^2)\sigma^{\mu\nu}p_{2\nu} + iA_5(q^2)\sigma^{\mu\nu}p_{4\nu}\right]u(2)$$

Using Dirac equation $(\gamma^\mu p_\mu - m)u = 0$ and $\bar{u}(\gamma^\mu p_\mu - m) = 0$ and the gauge invariance condition $q_\mu J^\mu = 0$

$q^\mu = p_4^\mu - p_2^\mu$ & $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

We get, $A_2 = A_3$ & $A_4 = -A_5$

Hence we obtain ,

$$\bar{u}(4)\Gamma^\mu u(2) = \bar{u}(4)[A_1\gamma^\mu + A_3(p_2 + p_4)^\mu + iA_5\sigma^{\mu\nu}(p_4 - p_2)_\nu]u(2) \text{ -------- } (7)$$

Continued...

Gordon Identity : $\bar{u}(4)\gamma^{\mu}u(2) = \frac{1}{2m}\bar{u}(4)[(p_4 + p_2\ )^{\mu} + i\sigma^{\mu\nu}(p_4 - p_2)_{\nu}]u(2)$ ------- (8)

Using Eq. (8), Eq. (7) becomes,

$\bar{u}(4)\Gamma^{\mu}u(2) = \bar{u}(4)[A_1\gamma^{\mu} + 2mA_3\gamma^{\mu} - iA_3\sigma^{\mu\nu}(p_4 - p_2)_{\nu} + iA_5\sigma^{\mu\nu}(p_4 - p_2)_{\nu}]u(2)$

$= \bar{u}(4)\left[F_1\gamma^{\mu} + \frac{i}{2M}F_2\sigma^{\mu\nu}(p_4 - p_2)_{\nu}\right]u(2)$ ----------- (9)

Redefining $F_1 = A_1 + 2mA_3$ and $\frac{F_2}{2M} = A_5 - A_3$

For simplicity using Eq. (8), we rewrite Eq. (9) as follows, (Replace m by M)

$\bar{u}(4)\Gamma^{\mu}u(2) = \bar{u}(4)\left[(F_1 + F_2)\gamma^{\mu} - \frac{1}{2M}F_2(p_4 + p_2)^{\mu}\right]u(2)$ ------------- (10)

From Eq. (1) , $\langle|M|^2\rangle = \frac{1}{4}\frac{g_e^4}{(p_1-p_3)^4}Tr[\gamma^{\mu}(\not{p}_1 + m)\gamma^{\nu}(\not{p}_3 + m)] \times Tr\left[\Gamma_{\mu}(\not{p}_2 + M)\Gamma_{\nu}(\not{p}_4 + M)\right]$

$Tr[\gamma^{\mu}(\not{p}_1 + m)\gamma^{\nu}(\not{p}_3 + m)] = 4\left[p_1^{\mu}p_3^{\nu} + p_3^{\mu}p_1^{\nu} + g^{\mu\nu}(m^2 - p_1 \cdot p_3)\right] \equiv L^{\mu\nu}$ ----------- (11)

For simplification $\Gamma^{\mu} = (F_1 + F_2)\gamma^{\mu} - \frac{1}{2M}F_2(p_4 + p_2)^{\mu} \equiv A\gamma^{\mu} + Bp^{\mu}$

Defining $A = F_1 + F_2$ $\quad B = F_2$ $\quad p = p_2 + p_4$

Continued…

$Tr\left[\Gamma_\mu(\not{p}_2+M)\Gamma_\nu(\not{p}_4+M)\right] = Tr\left[\left(A\gamma_\mu+Bp_\mu\right)(\not{p}_2+M)(A\gamma_\nu+Bp_\nu)(\not{p}_4+M)\right] \equiv H_{\mu\nu}$

Only traces containing odd number of γ matrices will be non-vanishing . On simplification, we get

$A^2 Tr\left[\gamma_\mu(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)\right] + B^2 p_\mu p_\nu Tr[(\not{p}_2+M)(\not{p}_4+M)] + ABp_\nu Tr\left[\gamma_\mu(\not{p}_2+M)(\not{p}_4+M)\right] + ABp_\mu Tr[(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)]$

$= A^2 Tr\left[\gamma_\mu(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)\right] + B^2 p_\mu p_\nu\{Tr[\not{p}_2\not{p}_4] + M^2 Tr[\mathbb{1}]\} + ABp_\nu\{MTr\left[\gamma_\mu\not{p}_2\right] + MTr\left[\gamma_\mu\not{p}_4\right]\} + ABp_\mu\{MTr[\not{p}_2\gamma_\nu] + MTr[\gamma_\nu\not{p}_4]\}$

- $A^2 Tr\left[\gamma_\mu(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)\right] = 4A^2\left[p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + g^{\mu\nu}(M^2 - p_2\cdot p_4)\right] = H^a_{\mu\nu}$ ------ (12)
- $B^2 p_\mu p_\nu\{Tr[\not{p}_2\not{p}_4] + M^2 Tr[\mathbb{1}]\} = 4B^2 p_\mu p_\nu[p_2\cdot p_4 + M^2] = H^b_{\mu\nu}$ ---------- (13)
- $ABp_\nu\{MTr\left[\gamma_\mu\not{p}_2\right] + MTr\left[\gamma_\mu\not{p}_4\right]\} = 4ABMp_\nu(p_2+p_4)_\mu$ ----- (c1)
- $ABp_\mu\{MTr[\not{p}_2\gamma_\nu] + MTr[\gamma_\nu\not{p}_4]\} = 4ABMp_\mu(p_2+p_4)_\nu$ ----- (c2)
- $H^c_{\mu\nu} = 8ABM(p_2+p_4)_\nu(p_2+p_4)_\mu$ (Adding c1 & c2) ------- (14)

$$\langle|M|^2\rangle = \frac{1}{4}\frac{g_e^4}{(p_1-p_3)^4}L^{\mu\nu}H_{\mu\nu} = \frac{1}{4}\frac{g_e^4}{q^4}L^{\mu\nu}H_{\mu\nu}$$

Continued…

$$L^{\mu\nu}H^{a}_{\mu\nu} = 64(F_1+F_2)^2M^2EE'\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \qquad \text{(Using Eq. (11) \& (12))}$$

$$L^{\mu\nu}H^{b}_{\mu\nu} = 4L^{\mu\nu}F_2^2(p_2+p_4)_\mu(p_2+p_4)_\nu[p_2\cdot p_4 + M^2] \qquad \text{(Using Eq. (11) \& (13))}$$

$$L^{\mu\nu}H^{c}_{\mu\nu} = 8L^{\mu\nu}(F_1+F_2)F_2M(p_2+p_4)_\nu(p_2+p_4)_\mu \qquad \text{(Using Eq. (11) \& (14))}$$

Elastic Scattering Kinematics :

$$p_1 \equiv (E,\vec{p}_1);\;\; p_2 \equiv (M,0);\;\;\; p_3 \equiv (E',\vec{p}_3)\;\;\&\;\; p_1+p_2 = p_3+p_4\;\;\&\; m_e \approx 0$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{1}{8\pi M}\right)^2\langle|M|^2\rangle = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left[\left(F_1^2 - \frac{q^2}{4M^2}F_2^2\right)\cos^2\frac{\theta}{2} - (F_1+F_2)^2\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \text{--------} \quad (15)$$

This is known as Rosenbluth Formula for Elastic $e-p$ scattering process.

- Two form factors $F_{1,2}(q^2)$ parametrize our ignorance about the detailed internal substructure of the proton represented by the blob in the FD of elastic $e-p$ scattering.
- These form factors can be determined experimentally by measuring $\frac{d\sigma}{d\Omega}$ as a function of θ and q^2.
- If proton were a point like Dirac fermion then Eq. (15) turns into Eq. (5) with $F_1(q^2) = 1$ and $F_2(q^2) = 0$ for all q^2.

Continued…

Instead of the functions $F_1(q^2)$ and $F_2(q^2)$ one often introduces the so-called electric and magnetic form factors denoted by $G_E(q^2)$ and $G_M(q^2)$ in such a way so that no interference term occurs in the cross-section.

$G_E(q^2) = F_1^2(q^2) - \frac{q^2}{4M^2} F_2^2(q^2)$ & $G_M(q^2) = F_1(q^2) + F_2(q^2)$

Now Eq. (15) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left[\frac{G_E^2(q^2)+\tau G_M^2(q^2)}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2(q^2) \sin^2\frac{\theta}{2}\right] \text{---------} \quad (16)$$

Where, the Lorentz invariant quantity, $\tau = -\frac{q^2}{4M^2}$

- The form factors $G_E(q^2)$ and $G_M(q^2)$ could be interpreted as the Fourier transforms of the charge and magnetic moment distributions of the proton. Unfortunately, the recoil of proton makes it impossible.
- However, it is possible to show that the form factors $G_E(q^2)$ and $G_M(q^2)$ are closely related to the proton charge and the magnetic moment distributions, respectively, in a particular Lorentz frame, called Breit frame, defined by $\vec{p}_4 = -\vec{p}_2$

Continued…

Scattering of an electron in a static potential due to an extended charge distribution:

The potential at $\vec{r}$ from the center is given by :

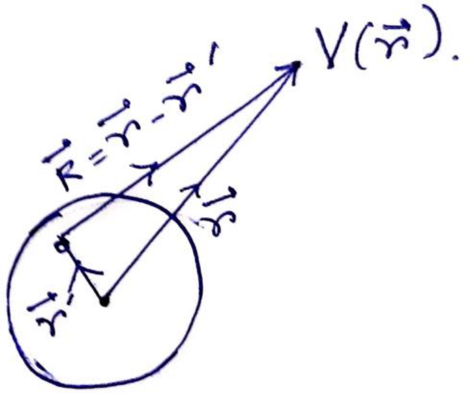
$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' \quad \text{with} \int \rho(\vec{r}) d^3\vec{r} = 1 \text{ (Normalization)}$$

Considering 1st order perturbation theory the matrix element is given by :

$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3\cdot\vec{r}} V(\vec{r}) e^{i\vec{p}_1\cdot\vec{r}}\, d^3\vec{r}$$

$$= \iint e^{i\vec{q}\cdot\vec{r}} \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' d^3\vec{r}$$

$$= \iint e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\, e^{i\vec{q}\cdot\vec{r}\,'} \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' d^3\vec{r}$$



Continued...

Where, $\vec{q} = \vec{p}_1 - \vec{p}_3$. Keeping $\vec{r}'$ fixed and integrate over $d^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

$$M_{fi} = \int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}')e^{i\vec{q}\cdot\vec{r}'} d^3\vec{r}' = (M_{fi})_{point} F(\vec{q})$$

Where $F(\vec{q})$ is Fourier transform of $\rho(\vec{r}')$. This resulting matrix element is equivalent to the matrix element of scattering from a point source multiplied by the form factor : $F(\vec{q}) = \int \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$

$$F(\vec{q}) = \int \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

If $|\vec{q}| \ll 1$, we can write $F(\vec{q}) = \int \left(1 + i\vec{q}\cdot\vec{r} - \frac{(\vec{q}\cdot\vec{r})^2}{2} + \cdots.\right) \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$

$= 1 - \frac{1}{6}|\vec{q}|^2\langle r^2\rangle + \cdots = F(|\vec{q}|^2)$ ------

Here, we have assumed $\rho(r)$ to be spherically symmetric, that is, a function of $r = |\vec{r}|$ only. The small-angle scattering therefore just measures the mean square radius $\langle r^2\rangle$ of the charge distribution. This is because in the small $|\vec{q}|$ limit the electron has large de'Broglie wavelength and can resolve only the size of the charge distribution $\rho(r)$ and is unable to probe the detailed structure.

Continued…

Rutherford scattering cross-section formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{\alpha^2}{16E_k^2\sin^4\frac{\theta}{2}}$$

This formula could have been derived by considering the scattering of a non-relativistic particle ($E_k = \frac{1}{2}mv^2$) in the static coulomb potential of the proton $V(\vec{r})$ without any consideration of the interaction due to intrinsic magnetic moments of electron or proton. Hence, we can conclude that in the non-relativistic limit only the interaction between electric charge of the particles matters.

In Rutherford scattering we consider the limit where the target recoil is neglected and the scattered particle is non-relativistic.

Mott scattering cross-section formula:

When we consider the recoil of the target to be neglected and the scattered particle is relativistic (i.e., the mass of electron being neglected), the scattering is called Mott scattering. ($E \approx E_k$ for $E \gg m$)

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2}$$

Overlap between initial/final state electron wave functions

Continued…

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \cos^2\frac{\theta}{2} |F(|\vec{q}|^2)|^2$$

There is nothing new in form factors – similar to diffraction of plane waves in optics. The finite size of scattering center introduces a phase difference between plane waves scattered from different points in space. If the wavelength is long copared to size of all waves in phase and $F(|\vec{q}|^2)$=1

NOTE that for point like charge the form factor is unity.

Point like Electron-Proton Scattering:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right) \frac{E'}{E}\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \text{(From eq. (5))}$$

$q^2 = -4EE' \sin^2\frac{\theta}{2}$; Note that $q^2 < 0$ i.e., Space like.

Show that $E - E' = -\frac{q^2}{2M}$

Since $q^2 < 0$; therefore, $E - E' > 0$ that is the scattered electron is always lower in energy than the incoming electron.

Continued…

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right)\frac{E'}{E}\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

How do we interpret the equation?

Compare with the Mott equation : $\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2}$; It is equivalent to scattering of spin ½ electron in a fixed electrostatic potential.

- The term $\left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right)$ is due to static charge distribution considering non-relativistic limit.
- The term $\frac{E'}{E}$ is due to recoil of proton.
- The term $\cos^2\frac{\theta}{2}$ is due to relativistic effect of the electron.
- The term $-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}$ is magnetic interaction due to the spin-spin interaction.

The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy of the scattered electron, E', are fixed by kinematics.

Continued…

Example:

From elastic scattering kinematics we can obtain the following two relations:

$$\frac{E'}{E} = \frac{M}{M+E(1-\cos\theta)} \quad \text{------- (a)} \quad \& \quad q^2 = \frac{2ME^2(1-\cos\theta)}{M+E(1-\cos\theta)} \quad \text{---------- (b)}$$

Let us consider $e-p$ scattering at $E_{beam} = 529.5 MeV$ and electrons scattered at an angle $\theta = 75^{\circ}$

For elastic scattering using Eq. (a) we get $E' = 373\ MeV$ and using Eq. (b) we get $q^2 = 294 MeV^2$

Elastic Scattering from a Finite Size Proton:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \frac{E'}{E}\left[\frac{G_E^2(q^2)+\tau G_M^2(q^2)}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2(q^2)\sin^2\frac{\theta}{2}\right]$$

Unlike our previous discussions of form factors, here the form factors are function of q^2 rather than $|\vec{q}|^2$and can not simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E-E')^2 - |\vec{q}|^2 = \frac{q^4}{4M^2} - |\vec{q}|^2 \Rightarrow -|\vec{q}|^2 = q^2\left[1-\left(\frac{q}{2M}\right)^2\right]$

For $\frac{q^2}{4M^2} \ll 1$ we have $-|\vec{q}|^2 \approx q^2$ and hence $G(q^2) \approx G(|\vec{q}|^2)$

Continued…

Hence in the limit $\frac{q^2}{4M^2} \ll 1$ we can interpret the structure functions in terms of the FT of the charge and magnetic moment distributions.

$$G_E(q^2) \approx G_E(|\vec{q}|^2) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

$$G_M(q^2) \approx G_M(|\vec{q}|^2) = \int \mu(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

Note that in deriving Rosenbluth formula we assumed proton to be spin ½ Dirac particle , i.e., $\vec{\mu} = \frac{e}{M}\vec{S}$. However, the experimentally measured value of proton magnetic moment is larger than that expected for a point-like Dirac particle: $\vec{\mu} = 2.79\frac{e}{M}\vec{S}$.

So for proton expect

$$G_E(0) = \int \rho(\vec{r}) d^3\vec{r} = 1 \qquad \& \qquad G_M(0) = \int \mu(\vec{r}) d^3\vec{r} = \mu_p = +2.79$$

It should be remembered that the anomalous magnetic moment of the proton is a strong evidence that it is not a point like particle !!

Continued…

How do we measure $G_E(q^2)$ & $G_M(q^2)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right]$$

Where

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \cos^2\frac{\theta}{2}$$

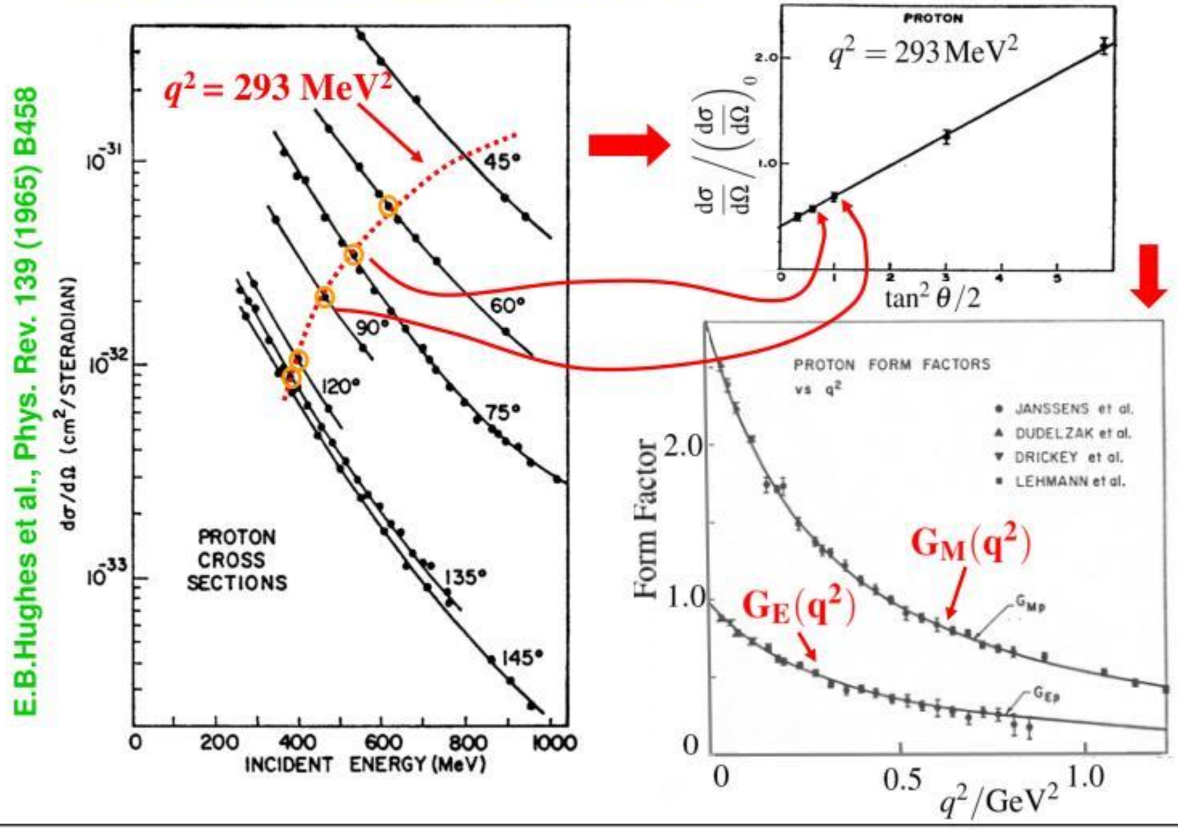
Mott scattering cross-section formula with recoil of proton. It corresponds to scattering from a spin-0 proton.

- At very low q^2: $\tau = \frac{q^2}{4M^2} \approx 0$ $\quad \therefore \left(\frac{d\sigma}{d\Omega}\right)_{lab} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2\,(q^2)$
- At high q^2: $\tau \gg 1 \therefore \left(\frac{d\sigma}{d\Omega}\right)_{lab} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2\frac{\theta}{2}\right) G_M^2\,(q^2)$
- In general from the intercept and slope of the plot $\left(\frac{d\sigma}{d\Omega}\right)_{lab} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0$ vs $\tan^2\frac{\theta}{2}$ one can estimate $G_E(q^2)$ & $G_M(q^2)$ provided q^2 is kept fixed.

Continued…

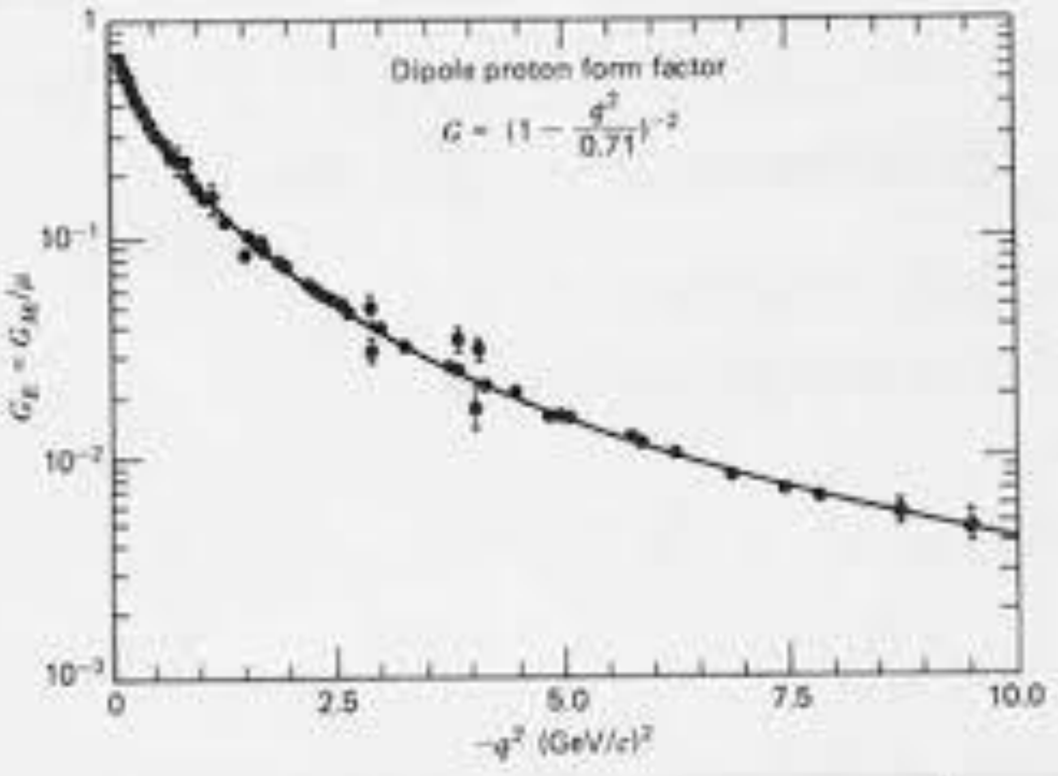
EXAMPLE: $e^-p \rightarrow e^-p$ at E_{beam}= 529.5 MeV

- Electron beam energies chosen to give certain values of q^2
- Cross sections measured to 2-3 %



NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution



Dipole proton form factor
$G = (1 - \frac{q^2}{0.71})^{-2}$
$G_E = G_M/\mu$
$-q^2$ (GeV/c)^2

Continued…

From the plots it is clear that form factors fall rapidly with the increase in q^2. For proton to be point like it would have been unity.

Conclusions:

- Proton is not point-like.
- Good fit to the data shows "dipole form factor" : $G_E^p(q^2) \approx \frac{G_M^p(q^2)}{2.79} \approx \left(1 - \frac{q^2}{0.71}\right)^{-2}$ (in units of GeV^2)
- Taking FT find spatial charge and magnetic moment distribution $\rho(r) \approx \rho_o e^{-\frac{r}{a}}$ with $a = 0.24\ fm$
- The rms charge radius is found to be $r_{rms} \approx 0.8\ fm$. It is obtained from $\langle r^2 \rangle = 6 \left(\frac{dG_E(q^2)}{dq^2}\right)_{q^2=0}$.
- Electron elastic scattering from proton demonstrates that the proton is an extended object wirh rms charge radius of $\sim 0.8\ fm$.

Continued…

- For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\right)\frac{E'}{E}\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

For elastic scattering of relativistic electrons from an extended proton:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\frac{E'}{E}\left[\frac{G_E^2(q^2) + \tau G_M^2(q^2)}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2(q^2)\sin^2\frac{\theta}{2}\right]$$

- Further probing of internal substructure of protons can be done by inelastic scattering of electrons from protons. That is done by increasing the energy of the incident electrons.