MODULE-5

- Inelastic electron-proton scattering
- Deep-inelastic scattering

Prepared by Sujoy Poddar
Department of Physics
Diamond Harbour Women’s University

Quantum Electrodynamics

Inelastic $e - p$ scattering

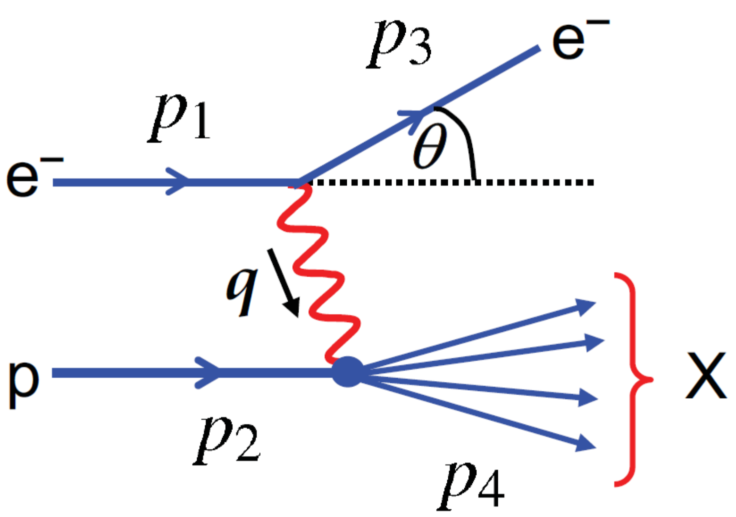
Elastic $e-p$ scattering at very high q^2

The Rosenbluth formula becomes

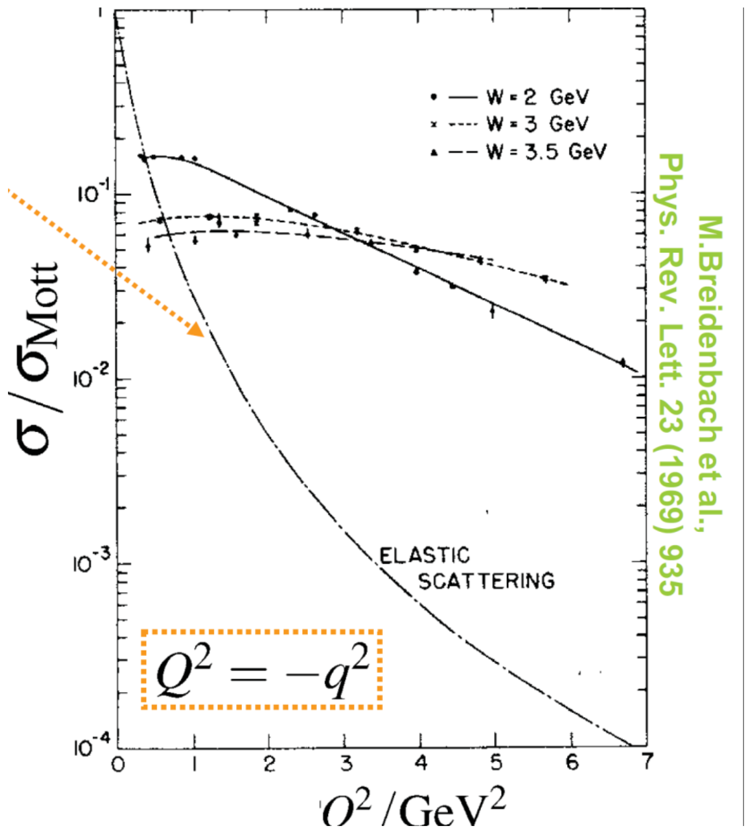
$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left[\frac{q^2}{2M^2}G_M^2(q^2)\sin^2\frac{\theta}{2}\right] \quad \text{where} \quad \tau = \frac{q^2}{4M^2} \ll 1$$

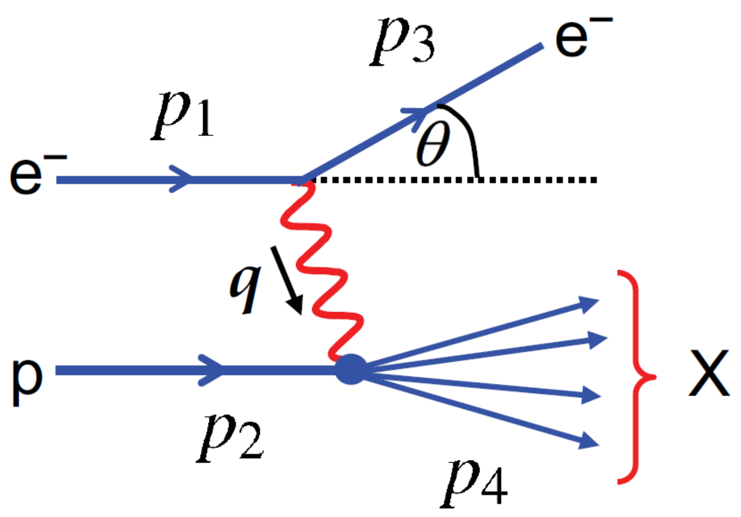
From $e-p$ scattering , the proton magnetic form factor is $\approx G_M(q^2) \approx \left(1-\frac{q^2}{0.71}\right)^{-2}$

At high q^2; $G_M(q^2) \propto q^{-4}$ hence, $\left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto q^{-6}$



Due to the finite proton size, elastic scattering at high q^2 is unlikely and inelastic reactions where the proton breaks up dominate.





- For inelastic scattering the mass of the final state hadronic is no longer the proton mass, M.
- The final hadronic state must contain at least one baryon which implies the final state invariant mass $M_X > M$

$$M_X^2 = p_4^2 = E_4^2 - |\vec{p}_4|^2$$

- For inelastic scattering introduce four Lorentz scalar kinematic variables : x, y, Q^2, ν

Define : $x \equiv \frac{Q^2}{2p_2 \cdot q}$; where $Q^2 \equiv -q^2$ and $Q^2 > 0$.

Here, $M_X^2 = W^2 = p_4^2 = (q + p_2)^2 = -Q^2 + 2p_2 \cdot q + M^2 \Rightarrow Q^2 = 2p_2 \cdot q + M^2 - M_x^2$

$\Rightarrow Q^2 \leq 2p_2 \cdot q$; Hence, $0 < x < 1$ [Inelastic] & $x = 1$ [Elastic] $[M_X = W = M]$ for intact proton

Define : y $\equiv \frac{p_2 \cdot q}{p_1 \cdot p_2}$; In Lab. $p_1 = (E, 0,0, E)$; $p_2 = (M, 0,0,0)$ & q $= (E - E', \vec{p}_1 - \vec{p}_3)$

Therefore, y $= \frac{\mathrm{E} - \mathrm{E}'}{E} = 1 - \frac{E'}{E}$; y measures the fractional energy loss of incoming electron

Hence, $0 < y < 1$

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Define : $\nu \equiv \frac{p_2 \cdot q}{M}$;

In the lab. frame $\nu = E - E'$; Energy lost by the incoming electron.

Relationships among the Four variables (x, y, Q^2, ν) :

Defining CM energy by $S = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \approx M^2 + 2p_1 \cdot p_2 \Rightarrow 2p_1 \cdot p_2 = S - M^2$

For fixed CM energy the four kinematic variables (Lorentz scalars) :x, y, Q^2, ν are not independent.

Hence, $x = \frac{Q^2}{2M\nu}$ $\quad y = \frac{2M}{S-M^2}\nu$ $\quad xy = \frac{Q^2}{S-M^2} \Rightarrow Q^2 = (S - M^2)xy$

For fixed CM energy, the interaction kinematics are completely defined by any two of the above kinematic variables. (except y and ν)

However, for elastic scattering $(x = 1)$ there is only one independent variable. It has already been seen that if scattering angle is measured everything else can be expressed in terms of that.

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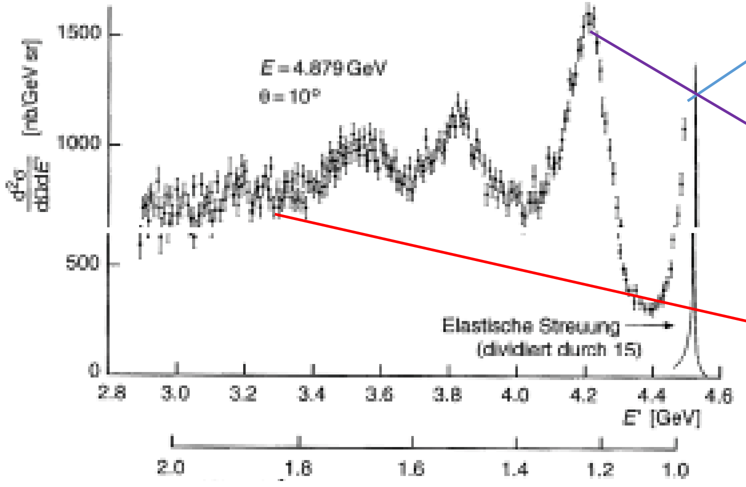
Inelastic Scattering:

Example: Scattering of 4.879 GeV electrons from protons at rest.

- Place the detector at 10° to the incoming electron beam and measure the energies of the scattered electrons.
- Kinematics can be fully determined from the electron energy and the angle of scattering - θ.
- The invariant mass of the final hadronic state is given by: $W^2 = M_X^2 = 10.06 - 2.03E'$. (Show this)



Elastic Scattering
$W = M$

Inelastic Scattering
$W = M_X$

DIS; Proton breaks into many particles & $DIS = Large\ W$

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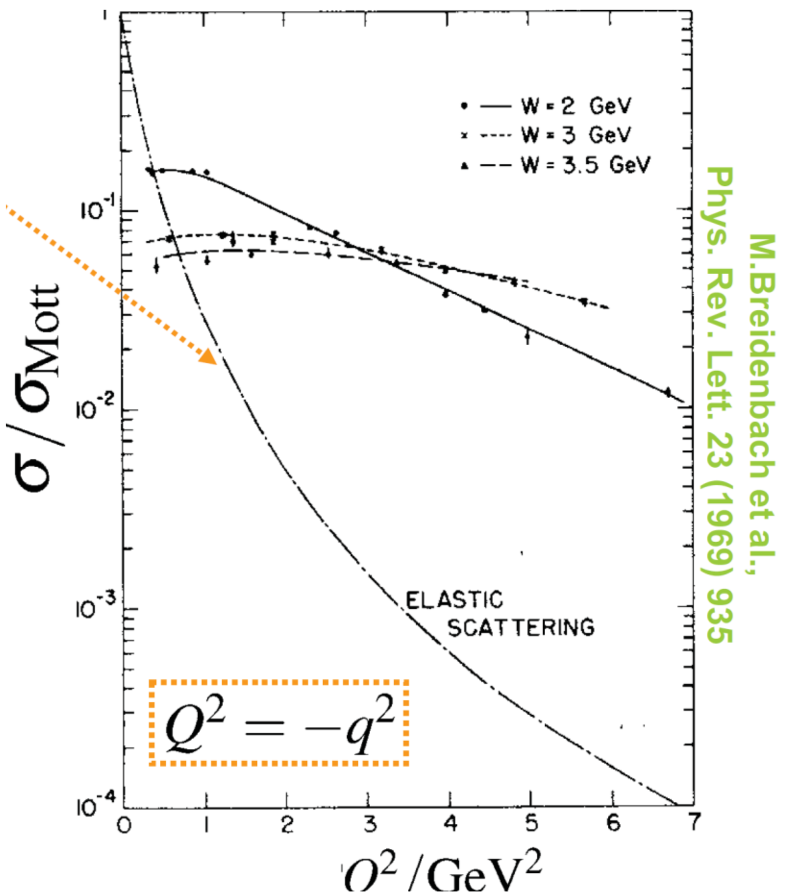

- Repeat the experiments at different angles/beam energies and determine the q^2 dependence of elastic and inelastic cross-sections.
- Elastic scattering falls of rapidly with q^2 due to proton not being point-like (Form factor)
- Inelastic scattering cross-sections weakly depends on q^2
- Deep Inelastic scattering cross-sections are fairly q^2 independent.
- In such case form factor is 1 => Scattering from a point like object within the proton is indicated.

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Recall Elastic Scattering :

In Lab. frame , the differential scattering cross-section can be expressed only in terms of electron scattering angle (Rosenbluth Formula)

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E}\left[\frac{G_E^2(q^2) + \tau G_M^2(q^2)}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2(q^2)\sin^2\frac{\theta}{2}\right]$$

Here the energy of the scattered electron is determined by the angle, θ.

In terms of Lorentz scalars the above differential cross-section can be expressed in terms of Q^2

$$\left(\frac{d\sigma}{dQ^2}\right)_{lab} = \frac{4\pi\alpha^2}{Q^4}\left[{\frac{G_E^2 + \tau G_M^2}{1+\tau}}^2\left(1 - y - \frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 G_M^2\right]$$

Can be rewritten as:

$$\left(\frac{d\sigma}{dQ^2}\right)_{lab} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1 - y - \frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 f_1(Q^2)\right]$$

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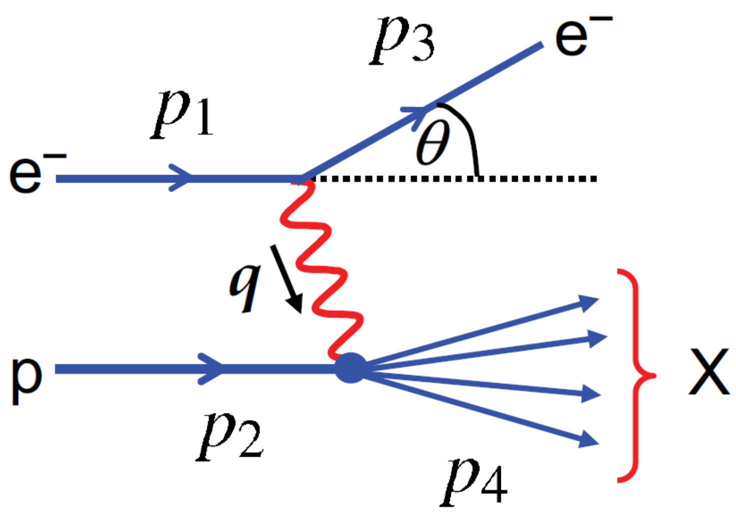

- Size of proton have been measured by elastic e-p scattering.
- Increasing q^2 of the photon would give us better spatial resolution i.e., more detailed structure of the proton.
- This is done by requiring a large energy loss of the bombarding e.
- Because of large energy transfer, sometimes proton breaks into many particles.
- For modest q^2 , one might just excite the proton into a Δ -state and produce an extra π meson, that is $ep \to e\Delta^+ \to ep\pi^0$. Here the invariant mass $W^2 = M_\Delta^2$
- Further increase in q^2 results in breaking of proton into multiparticles.

In case inelastic $e - p$ scattering the proton-photon vertex current can not be written like elastic case since proton after being bombarded by electrons produce multiparticles and therefore can not be described by single Dirac spinor $\bar{u}$.

In stead we can deal with the 2nd rank hadron tensor($W^{\mu\nu}$) like 2nd rank lepton tensor($L_{\mu\nu}$).

Let us construct the $W^{\mu\nu}$: Out of $g^{\mu\nu}$; two indepenent 4-momenta q and $p_2 (= p)$. γ^μ not included, Why?

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The most general form of $W_{\mu\nu}$:

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p_2^\mu p_2^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2}\left(p_2^\mu q^\nu + q^\mu p_2^\nu\right)$$

Note that W_3 is omitted in the above expression which contains parity violating terms - γ_5. This will be included when neutrino beam will be considered in stead of electron beam and virtual photon will be replaced by weak bosons.

Prob: Show that $L_{\mu\nu} = L_{\nu\mu}$ and $q^\mu L_{\mu\nu} = 0$. [Consider Eq. (11)]

Prob: Show that current conservation at the hadronic vertex requires $q_\mu W^{\mu\nu} = 0$. Hence, verify that

$W_5 = -\frac{p\cdot q}{q^2} W_2$ & $W_4 = \left(\frac{p\cdot q}{q^2}\right)^2 W_2 + \frac{M^2}{q^2} W_1$.

Also show that $W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right)$ ------------ (17)

Using Eq. (11) & (17) we get

$$L_{\mu\nu} W^{\mu\nu} = 4W_1(p_1\cdot p_3) + \frac{2W_2}{M^2}\left[2(p_2\cdot p_1)(p_2\cdot p_3) - M^2 p_1\cdot p_3\right]$$

In Lab. frame, this becomes

$L_{\mu\nu} W^{\mu\nu} = 4EE'\left[\mathrm{W}_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2}\right]$ [Check!] --------------------------- (18)

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Now the coefficients W_1, W_2 are not only function of q^2 but also function of ν . Now $p_4^2 \neq M^2$. Moreover, W_1, W_2 can no longer be interpreted as the Fourier transforms of the charge and magnetic moment distributions. They are called structure functions which describe momentum distribution of the partons within the proton.

Master formula for 2->2 scattering ($e\, p \;\rightarrow e\, X$) cross-section (Golden Rule):

$$d\sigma = \frac{1}{4\left((p_1 \cdot p_2)^2 - m^2 M^2\right)^{\frac{1}{2}}} \left[\frac{g_e^4}{q^4} L_{\mu\nu} W^{\mu\nu} 4\pi M\right] \frac{d^3 p_3}{2E'(2\pi)^3}$$

$$\frac{d\sigma}{dE' d\Omega} = \frac{\alpha^2}{q^4} \frac{E'}{E} L_{\mu\nu} W^{\mu\nu} \quad ;\text{using } d^3 p_3 = |\vec{p}_3|^2 d|\vec{p}_3| d\Omega = E'^2 dE' d\Omega$$

Therefore, in Lab. frame we obtain,

$$\left.\frac{d^2\sigma}{dE' d\Omega}\right|_{lab} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\mathrm{W}_2(\nu, q^2) \cos^2 \frac{\theta}{2} + 2W_1(\nu, q^2) \sin^2 \frac{\theta}{2}\right] - - - - - - - (19)$$

Verify the above expression

This is inclusive differential cross-section for the inelastic $e - p$ scattering, $e\, p \;\rightarrow e\, X$

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In terms of Lorentz invariant variables the inelastic cross-section can be expressed as :

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{F_2(x,Q^2)}{x}\left(1 - y - \frac{M^2x^2y^2}{Q^2}\right) + y^2F_1(x,Q^2)\right] ------(20)$$

Where $x = \frac{Q^2}{2M\nu}$; $\nu = E - E'$ & $\mathrm{y} = \frac{\mathrm{E-E'}}{E} = 1 - \frac{E'}{E} = \frac{\nu}{E}$

In the limit of high energy ($Q^2 \gg M^2y^2$), the above equation becomes :

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right] -------(21)$$

In the Lab. frame , the above expression can be rewritten in terms of two variables θ, E' (experimentally measured quantities)

$$\left.\frac{d^2\sigma}{dE'd\Omega}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[\mathrm{W}_2(\nu,q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu,q^2)\sin^2\frac{\theta}{2}\right] ----(22)$$

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$$\frac{d^2\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\{\quad\} - - - - -(23)$$

For a muon target of mass m the above curly bracketed term is,

$$\{\quad\}_{e\mu\to e\mu} = \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2m}\right) - - - - - - -(24)$$

For elastic scattering from a proton target,

$$\{\quad\}_{ep\to ep} = \left(\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2M}\right) - - - - - - -(25)$$

Where M is the mass of the proton and $\tau = -\frac{q^2}{4M^2}$

Finally, for the case where proton target is broken up by the bombarding electron,

$$\{\quad\}_{ep\to eX} = \mathrm{W}_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2} - - - - -(26)$$

If one increases the energy of the bombarding electron sufficiently, the electron starts interacting with a point like Dirac particle inside the proton called "PARTON". This regime of energy is known as Deep Inelastic Scattering (DIS) regime.

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$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad \delta(-x) = \delta(x) \qquad E' = \frac{E}{1+\frac{2E}{M}\sin^2\frac{\theta}{2}} = \frac{1}{A}\,(Let)$$

$$\delta\left(\nu - \frac{Q^2}{2M}\right) = \delta\left(E - E' - \frac{4EE'\sin^2\frac{\theta}{2}}{2M}\right) = \delta\left[E - E'\left(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\right)\right] = \delta(E - AE') = \frac{1}{A}\delta\left(E' - \frac{E}{A}\right)$$

From Eq. (23) & (24) (Considering e-muon scattering)

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2m}\right)$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{Aq^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)\delta\left(E' - \frac{E}{A}\right)$$

Performing the integration using Dirac delta function, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)$$

Using $q^2 = -4EE'\sin^2\frac{\theta}{2}$ and substituting the expression of A

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For large $-q^2\ (= Q^2)$, i.e., in the deep inelastic regime, suddenly virtual photons "see" the indivisible component of the proton, i.e., called partons (quarks). For large $-q^2$, the existence of structureless particles inside the proton can be understood by observing the fact that suddenly proton behaves like point like free Dirac particle described by Eq. (26) turns into Eq. (24).

$$\frac{d^2\sigma}{dE' d\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left(\mathrm{W}_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2}\right)$$

$$\frac{d^2\sigma}{dE' d\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2m}\right)$$

Thus comparing the above two equations proton structure functions can be expressed as :

$$2W_1^{point} = \frac{Q^2}{2m^2}\delta\left(\nu - \frac{Q^2}{2m}\right)$$

$$W_2^{point} = \delta\left(\nu - \frac{Q^2}{2m}\right) ------- -(27)$$

The "point" notation tells us that the quark is structureless Dirac particle having mass m

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At large Q^2, inelastic electron-proton scattering can be viewed as elastic scattering of the electron on a free quark inside the proton. Let us introduce dimensionless structure functions :

$$2mW_1^{point}(\nu, Q^2) = \frac{Q^2}{2m\nu}\delta\left(\nu - \frac{Q^2}{2m\nu}\right)$$

$$\nu W_2^{point} = \delta\left(\nu - \frac{Q^2}{2m\nu}\right) ------ -(28)$$

Using $\delta\left(\frac{x}{a}\right) = a\delta(x)$ we obtain the above expressions. Now it is evident that the structure functions are not functions of two independent variables ν and Q^2 but only functions of $\frac{Q^2}{2m\nu}$. This behaviour can be contrasted with that for ep elastic scattering. Assuming $G_E = G_M \equiv G$; then, comparing Eqs. (25) & (26), we have

$$W_1^{elastic} = \frac{Q^2}{4M^2}G^2(Q^2)\delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$W_2^{elastic} = G^2(Q^2)\delta\left(\nu - \frac{Q^2}{2M}\right) ---- -(29)$$

Note the difference between Eq. (27) & (29). Hence, can not be rearranged in terms of a single dimensionless variable.

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- Elastic form factors contain mass scale and it is set by the empirical value 0.71 GeV in the dipole formula for $G(Q^2)$which reflects the inverse size of the proton.
- As Q^2 increases above $(0.71GeV)^2$, the form factor depresses the chance of elastic scattering and the proton is more likely to break up.

Summarizing the above results we have

For large Q^2 virtual photons resolve point constituents inside the proton, then

$$MW_1(\nu, Q^2) \to F_1(x)$$
$$\nu W_2(\nu, Q^2) \to F_2(x)$$

Where $x = \frac{Q^2}{2M\nu}$ with $\nu = E - E'$

It is observed in SLAC that the structure functions are independent of Q^2 for fixed x. It establishes the fact that free quarks exist inside the proton. This is equivalent to the onset of $\frac{1}{\sin^4\frac{\theta}{2}}$ behavior for large momentum transfer in the Rutherford experiment which reveals the fact that atom has point charged nucleus.

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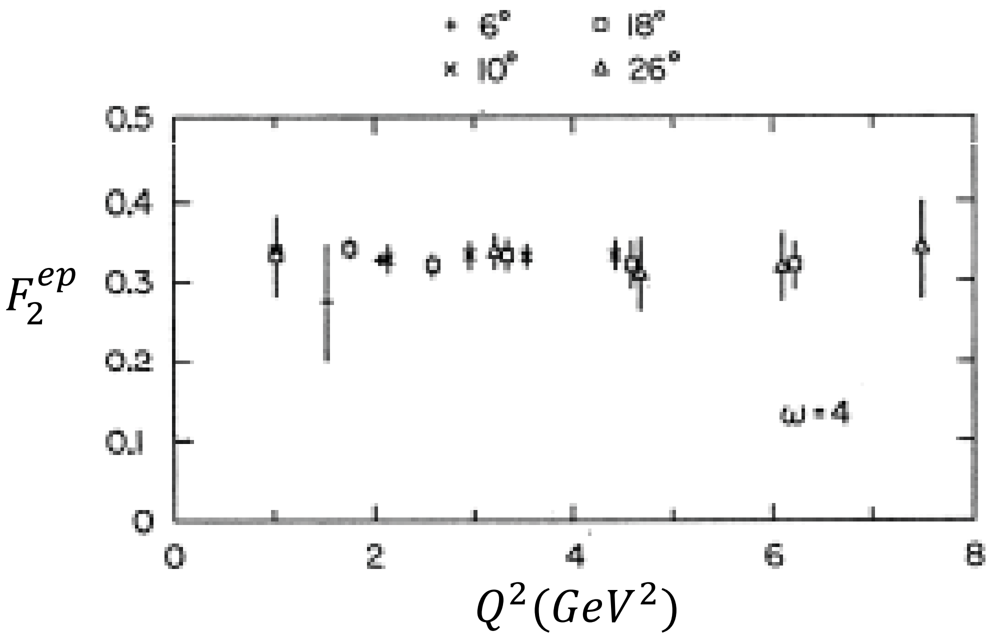

- Experimentally it has been observed that both F_1 and F_2 are almost independent of Q^2.
- This x is known as Bjorken scaling parameter which is valid in DIS regime.
- It is strongly suggestive that electrons are scattered by point like constituents inside the proton.
- It is also observed that F_1 and F_2 are not independent and they are related by Callan-Gross relation $F_2(x) = 2xF_1(x)$
- This relation shows that partons arte spin ½ particles. If partons were spin 0 particles, F_1 would have been 0, i.e., magnetic form factors is zero.

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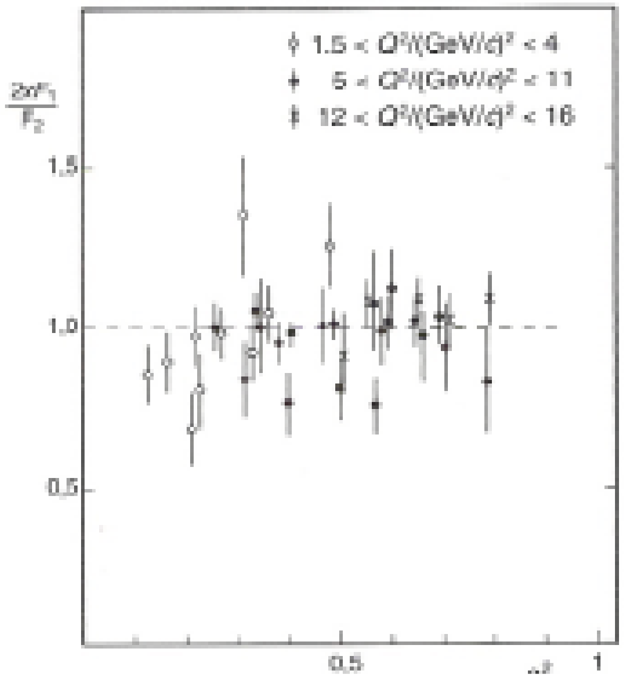

Along y-axis $\frac{2xF_1}{F_2}$ is plotted and it is seen that it is around 1 . Hence electrons are scatted by spin ½ quarks. The value would have been 0 if electrons get scattered by spin 0 structureless particles.

These structure functions can be interpreted as momentum distribution functions of quarks inside the proton which will be shown in Quark-Parton model.

In terms of structure functions F_1 & F_2 , the differential scattering cross-section looks like

$$\left.\frac{d^2\sigma}{dE'd\Omega}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[\frac{1}{\nu}F_2(\nu,Q^2)\cos^2\frac{\theta}{2} + \frac{2}{M}F_1(\nu,Q^2)\sin^2\frac{\theta}{2}\right]$$

In terms of Lorentz invariant variables the differential equation can be written into the following form

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{F_2(x,Q^2)}{x}\left(1-y-\frac{M^2x^2y^2}{Q^2}\right) + y^2F_1(x,Q^2)\right]$$

The structure functions can be determined for fixed x and Q^2 and it requires measurements of the differential scattering cross-sections at several different scattering angles and incoming electron beam energies.

Problem: Arrive at 2nd equation starting from the 1st one.

In the limit of high energy ($Q^2 \gg M^2y^2$ & $x \leq 1$), the above equation becomes :

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right]$$