MODULE-2

- Nuclear Shell Model
- Determination of Spin-Parity of ground state
- Magnetic dipole moment and Quadrupole moment
- Excited states of nucleus

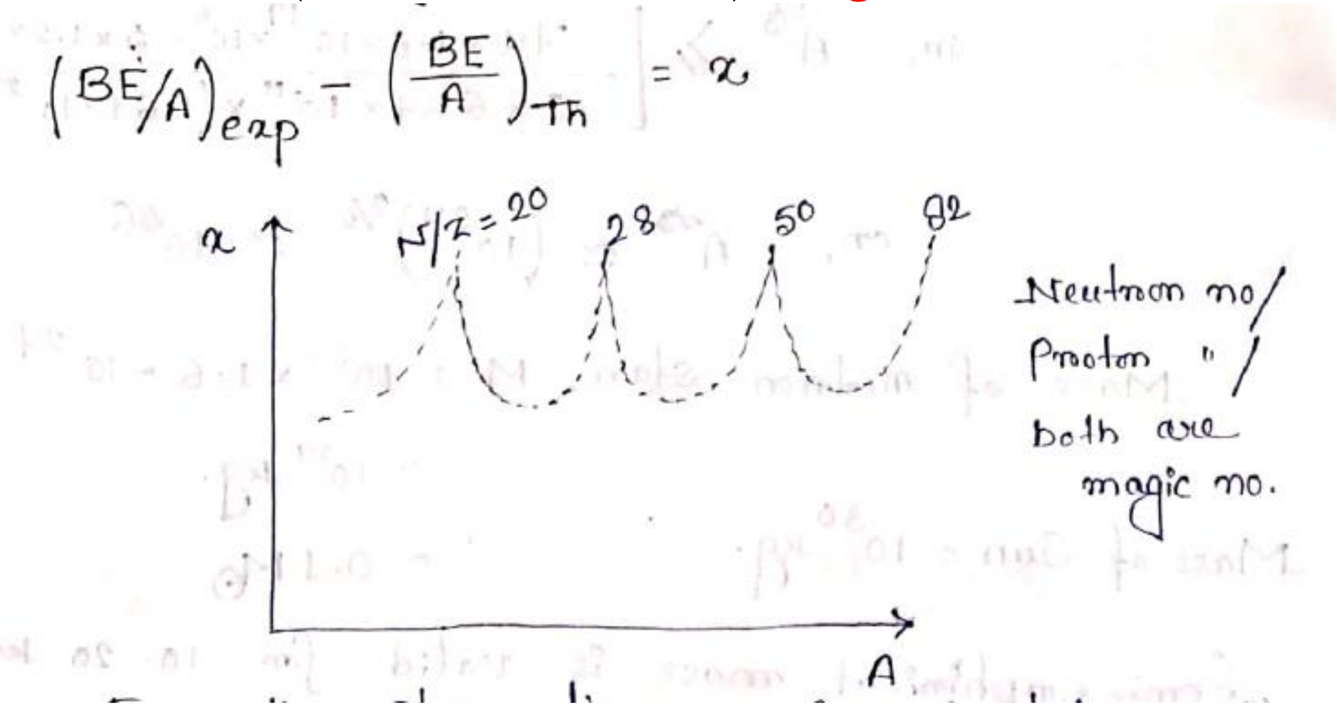
Prepared by Sujoy Poddar
Department of Physics
Diamond Harbour Women’s University

NUCLEAR MODELS(L1)

Nuclear Shell Model:

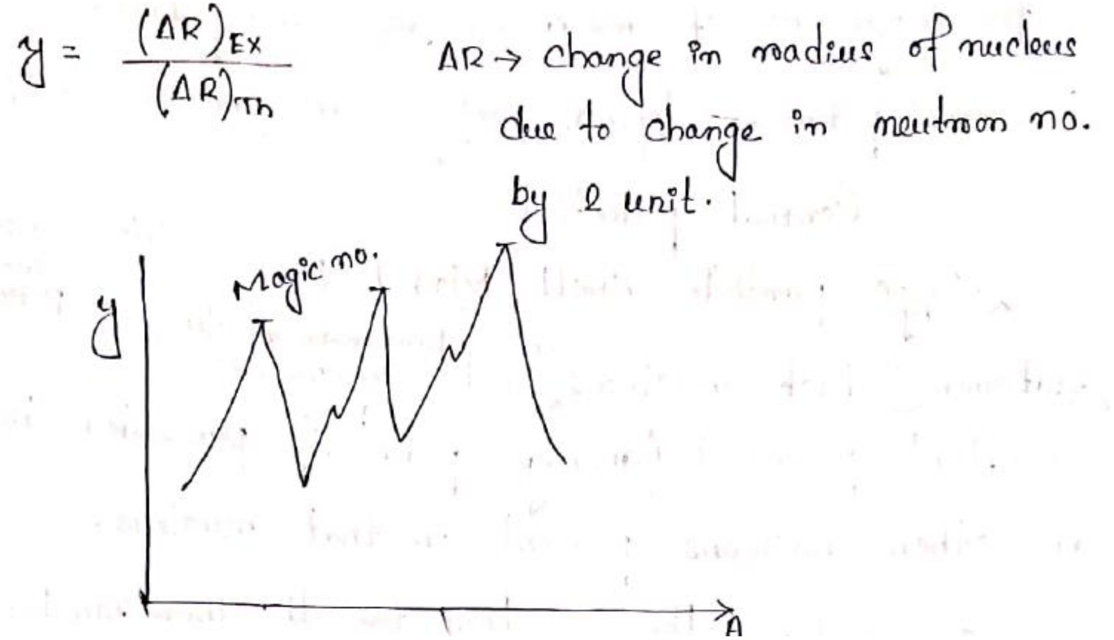
Evidences that support that Nucleus has shell structure:

1. Like Inert Gases which exhibit extra stability due to particular values of atomic number in Atomic Physics there are nuclei which show extra degree of stability for particular values of either proton number or neutron number or the both. (2,8,20,28,50,82,126), Magic numbers.



2. If x as defined in the adjacent figure is plotted against A, we observe that there are sharp spikes for N/Z=magic numbers.

3.



The above plot shows that the measured quantity, y exhibits sharp peaks at particular values of A which contains either magic number of neutrons or protons or both.

4. Neutron separation energy/Proton separation energy for nuclei containing magic number of neutrons/protons is large compared to other nuclei.

5. Neutron absorption cross-section for nuclei containing magic number of neutrons is very low compared to others.

6. The number of naturally occurring isotopes and isotones are greater for nuclei containng magic number of N or Z

Single Particle Shell Model:

Since each nucleus comprises many nucleons, hence it is a many body problem difficult to handle in quantum mechanics.

Assumption: Each nucleon in a particular nucleus experiences an average potential produced by the other nucleons present in that nucleus. Nucleons being indistinguishable particles, this is same for any nucleon. Hence each nucleon would feel the same average potential due to the remaining nucleons.

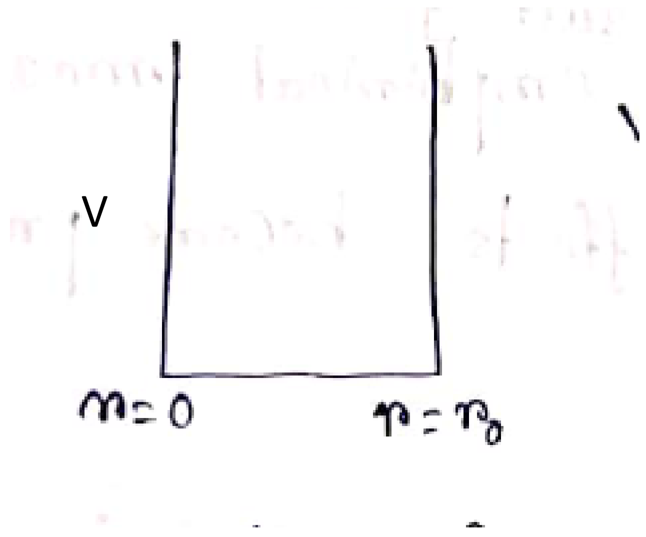
Question: What would be the form of this potential? Is it a central potential??

First choice is to play with central potential. There are many choices of central potentials like, 3D infinite square well, 3D HO, Wood-Saxon potential.

Infinite 3D square well potential.

$$V = 0; \qquad r < r_0$$

$$= -V_0; \qquad r > r_0$$



$$\Psi(r,\theta,\phi) = \frac{u(r)}{r} Y_m^l(\theta,\phi)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[V(r) + \frac{l(l+1)}{2mr^2}\right]u(r) = Eu(r)$$

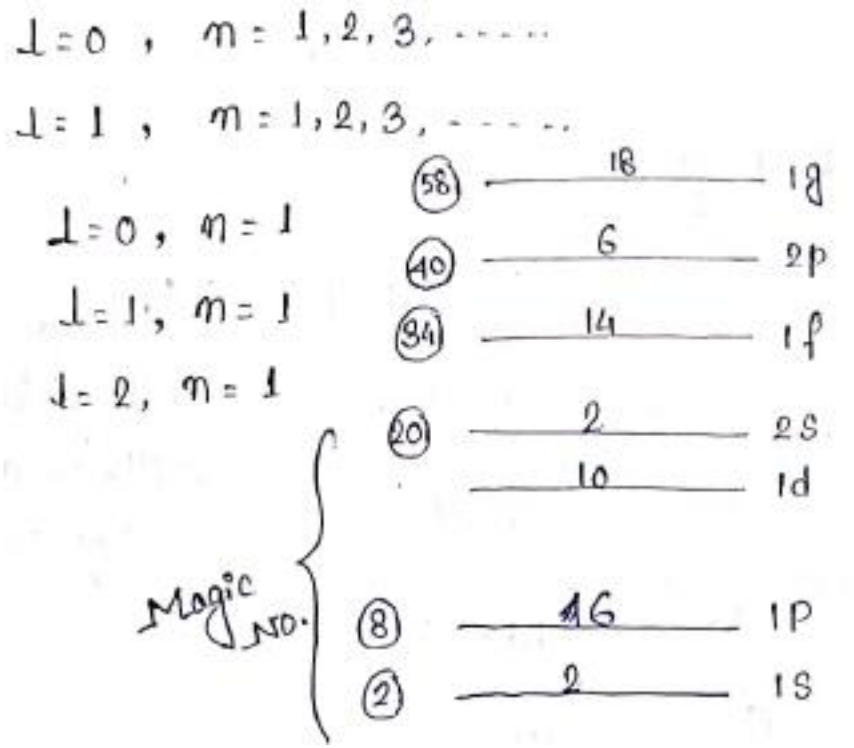
$$r < r_o; \qquad V = 0$$

Solution: $u(r) = rj_l(kr); \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

At $r = r_0, \quad j_l(kr_0) = 0$

For different values of l we have different discrete energy levels.

$$For\ \ l = 0$$
$$j_0(kr) = 0$$
$$\frac{\sin kr}{kr} = 0 \Rightarrow kr = n\pi$$
$$n = 1,2,3, \ldots$$



- Energy levels are found to be descrete.
- The degeneracy of each energy level is $2(2l + 1)$
- Using this form of potential upto $N\ or\ Z = 20$ can be explained but this form of potential fails to explain the other higher magic numbers.

3D Harmonic Oscillator Potential:

$$V(r) = \frac{1}{2}m\omega^2 r^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[\frac{1}{2}m\omega^2 r^2 + \frac{l(l+1)}{2mr^2}\right]u(r) = Eu(r)$$

$$E = \left(2n + l + \frac{3}{2}\right)\hbar\omega$$

$$= \left(N + \frac{3}{2}\right)\hbar\omega$$

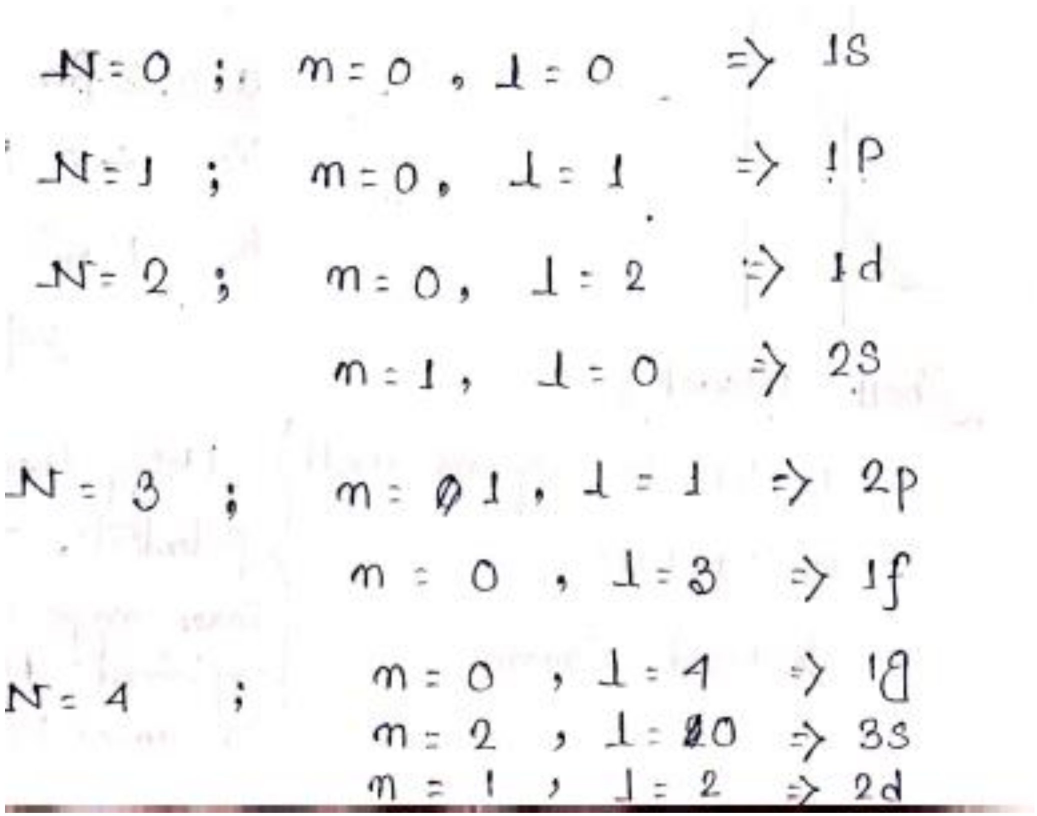
$$N = 0,1,2,,,,,,$$

$$n = 0,1,2,3 \ldots$$

$$l = 0,1,2,3 \ldots..$$

For notational convenience, we define $n' = n + 1$

Therefore the energy states will be $1s$ for $n = 0$ or $n' = 1$.Similarly for other energy states we will follow these convensuions.



N = 0 ; n = 0 , l = 0 ⇒ 1S

N = 1 ; n = 0 , l = 1 ⇒ 1P

N = 2 ; n = 0 , l = 2 ⇒ 1d

n = 1 , l = 0 ⇒ 2S

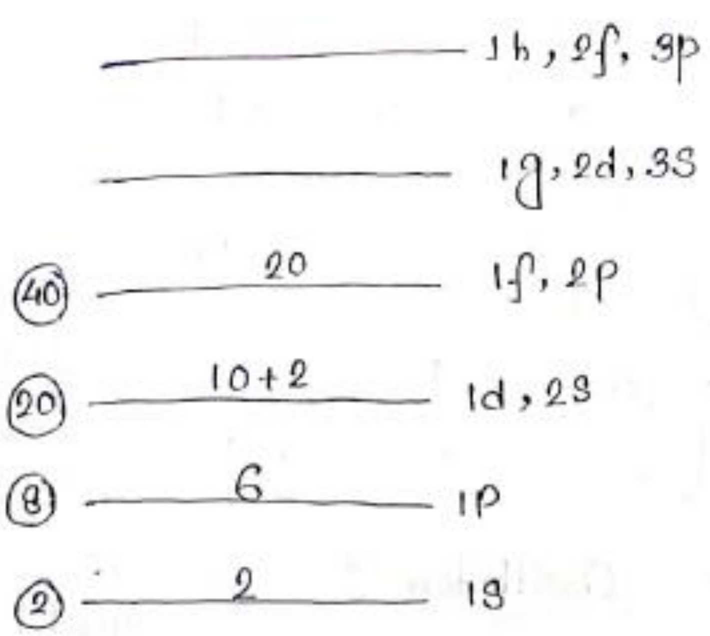
N = 3 ; n = 1 , l = 1 ⇒ 2p

n = 0 , l = 3 ⇒ 1f

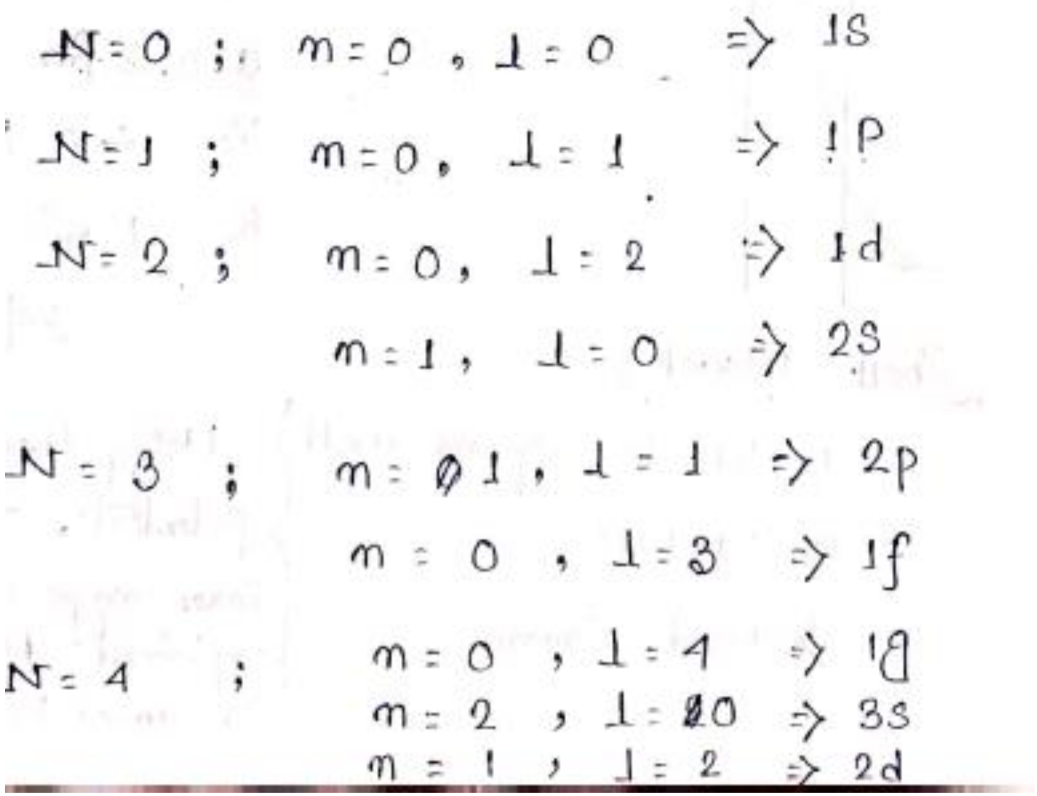
N = 4 ; n = 0 , l = 4 ⇒ 1g

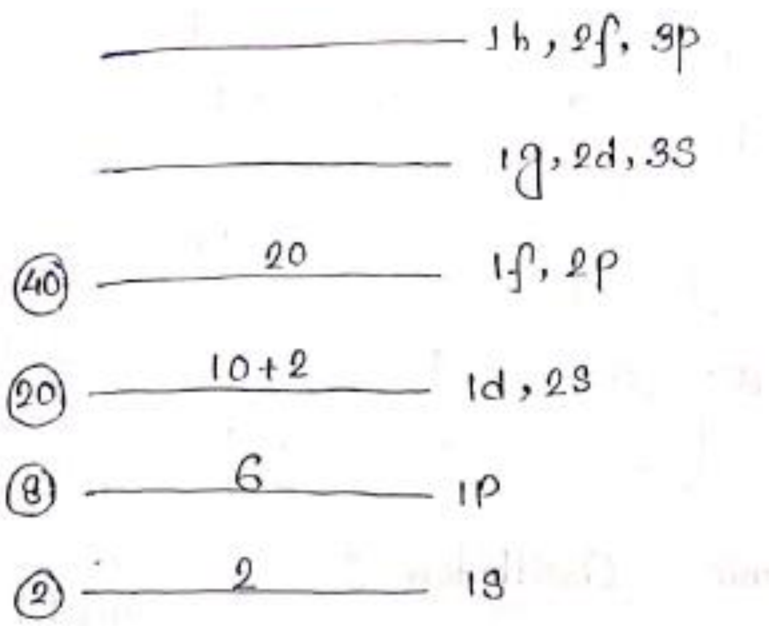
n = 2 , l = 0 ⇒ 3S

n = 1 , l = 2 ⇒ 2d



- The two forms of potentials considered so far are of infinite nature.
- Therefore energy needed for pulling out one nucleon is infinite.
- The neutron/proton separation energies are of finite amount already observed by experiments.
- Finite potentials should be considered for realistic situations.





- The two forms of potentials considered so far are of infinite nature.
- Therefore energy needed for pulling out one nucleon is infinite.
- The neutron/proton separation energies are of finite amount already observed by experiments.
- Finite potentials should be considered for realistic situations.

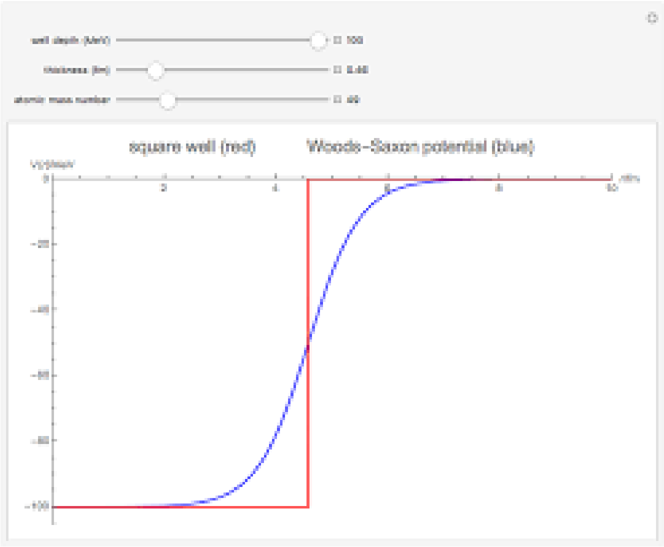
Wood-Saxon Potential

Form: $V(r) = -\frac{V_0}{1+e^{(r-R)/a}}$

$V_0 \approx 50\ MeV$

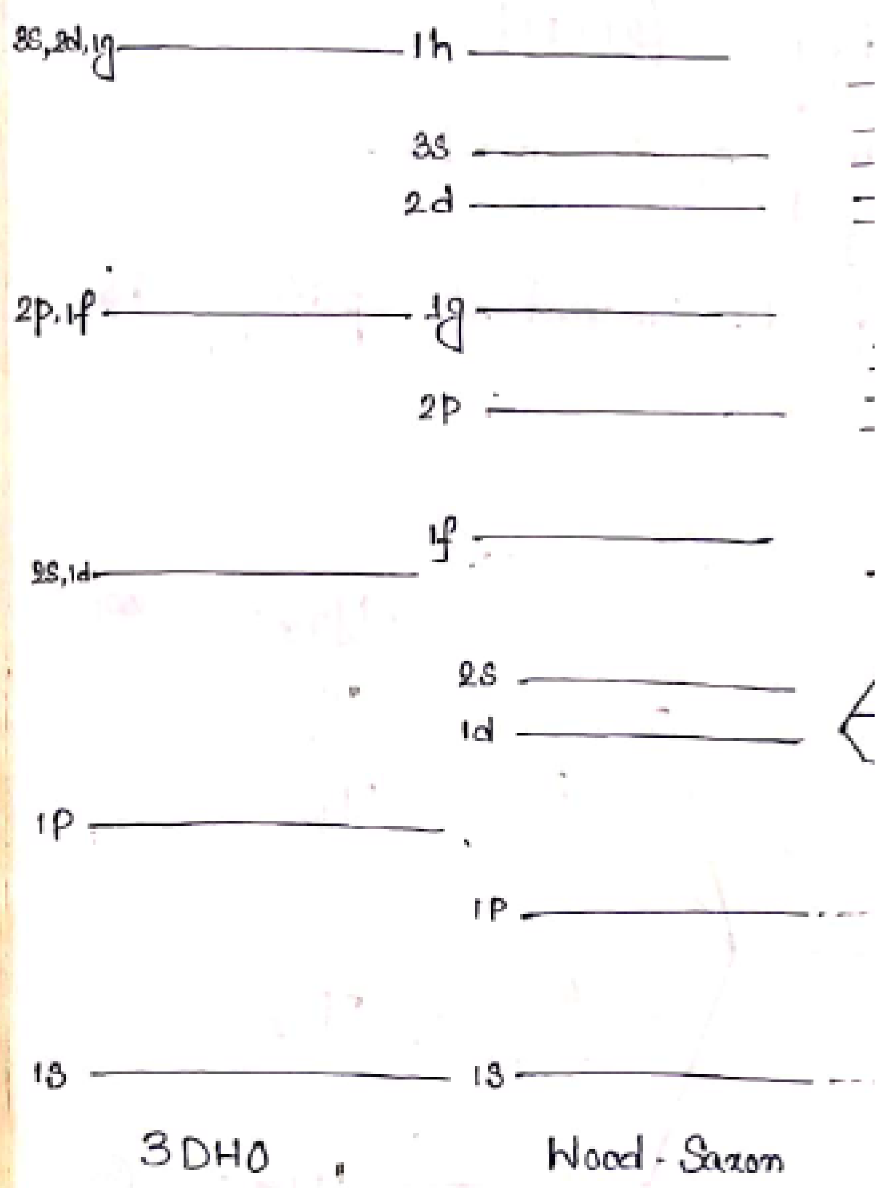
$a = 0.52\ fm$

$R = R_0 A^{1/3}$



Features:

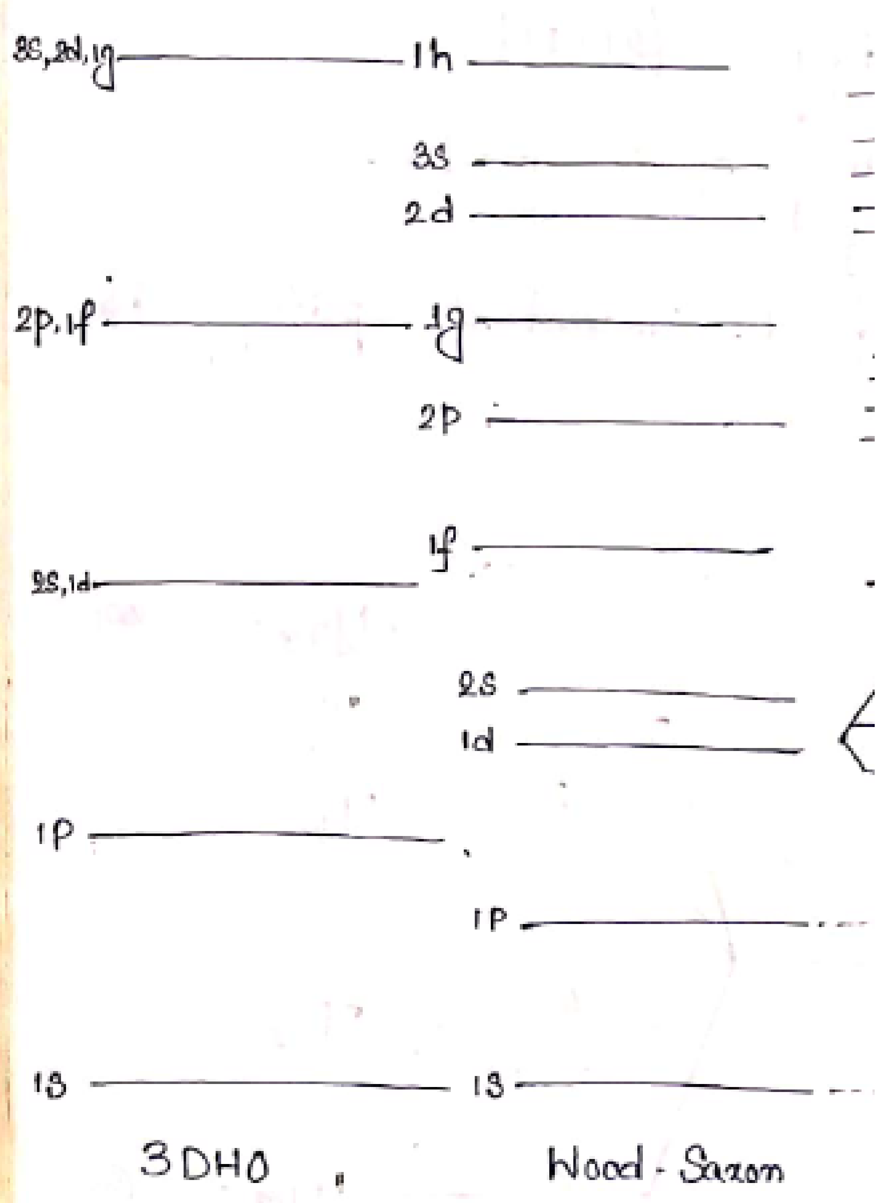
- Potential has finite depth
- Parameters are fixed by experimental results.
- Potential contains mass number A , hence for the tapered region it would depend on the size of the nucleus.
- For higher mass numbers the falling region of the potential extends over a finite region.


3s,2d,1g
1h
3s
2d
2p,1f
1g
2p
1f
2s,1d
2s
1d
1p
1p
1s
1s
3DHO
Wood - Saxon

- All the three potentials can explain magic numbers upto 20.
- Even the finite potential does not explain the higher magic numbers. It only shifts the energy levels which were equi-spaced for 3D HO and the degeneracy of the energy levels were lifted.
- In all three central potentials spin-orbit coupling is not considered. Consideration of this term in the potential make it non-central.
- Can it rescue us from this problem?

  Consider the Hamiltonian : $H = H_0 + f(r)\vec{l}\cdot\vec{s}$
- $f(r) < 0$; l and s denote orbital and spin angular momentum quantum numbers respectively.
- H commutes with L^2, S^2 *but not with* L_Z, S_Z
- $[H, L^2] = 0;\ [H, S^2] = 0;\ [H, L_Z] \neq 0;\ [H, S_Z] \neq 0$
- Hence, L_Z and S_Z do not have simultaneous eigenstates.
- Spin up and down states get mixed up and different m_l - value states will also be mixed up.
- If one considers $\vec{J} = \vec{L} + \vec{S}$, then $[H, J^2\ ] = [H, J_Z] = 0$. Thus eigenstates of the Hamiltonian would be $|l, s, j, m_j\ \ >$. $J^2|{**}> = j(j+1)\hbar^2|\ {**}>$ and $J_Z\big|{**}> = m_j\hbar\big|\ {**}>$; $m_j = -j\ \ldots . +j$ with steps of unity.
- $m_j = m_l + m_s$; different combinations of m_l and m_s can give rise to same m_j

NUCLEAR MODELS(L2)


3s, 2d, 1g
1h
3s
2d
2p, 1f
1g
2p
1f
2s, 1d
2s
1d
1p
1p
1s
1s
3DHO
Wood - Saxon

- All the three potentials can explain magic numbers up to 20.
- Even the finite potential does not explain the higher magic numbers. It only shifts the energy levels which were equispaced for 3D HO and the degeneracy of the energy levels were lifted.
- In all three central potentials spin-orbit coupling is not considered. Consideration of this term in the potential make it non-central.
- Can it rescue us from this problem?

Consider the Hamiltonian : $H = H_0 + f(r)\vec{l} \cdot \vec{s}$

- $f(r) < 0$; l and s denote orbital and spin angular momentum quantum operators respectively.
- H commutes with l^2, s^2 *but not with* l_Z, s_Z
- $[H, l^2] = 0;\ [H, s^2] = 0;\ [H, l_Z] \neq 0;\ [H, s_Z] \neq 0$
- Hence, l_Z and s_Z do not have simultaneous eigenstates.
- Spin up and down states get mixed up and different m_l - value states will also be mixed up.
- If one considers $\vec{j} = \vec{l} + \vec{s}$, then $[H, j^2\ ] = [H, j_Z] = 0$. Thus eigenstates of the Hamiltonian would be $|l, s, j, m_j\ >$. $j^2|{**}> = j(j+1)\hbar^2|\ {**}>$ and $j_Z|{**}> = m_j\hbar|\ {**}>$; $m_j = -j \ldots. +j$with steps of unity.
- $m_j = m_l + m_s$; different combinations of m_l and m_s can give rise to same m_j

$$H = H_0 + f(r)\vec{l}\cdot\vec{s}$$

$$\vec{j} = \vec{l} + \vec{s}$$

$$j^2 = l^2 + s^2 + 2\vec{l}\cdot\vec{s}$$

$$\langle\vec{l}\cdot\vec{s}\rangle = \frac{\langle j^2 - l^2 - s^2\rangle}{2}$$

- Since $s = \frac{1}{2}$ therefore, $j = l \pm \frac{1}{2}$

  Consider the case for $j = l + 1/2$

$$\langle\vec{l}\cdot\vec{s}\rangle = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$$

$$\langle\vec{l}\cdot\vec{s}\rangle = \frac{\hbar^2}{2}\left[(l+\frac{1}{2})\left(l+\frac{3}{2}\right) - l(l+1) - \frac{3}{4}\right]$$

$$\langle\vec{l}\cdot\vec{s}\rangle = \frac{l\hbar^2}{2}$$

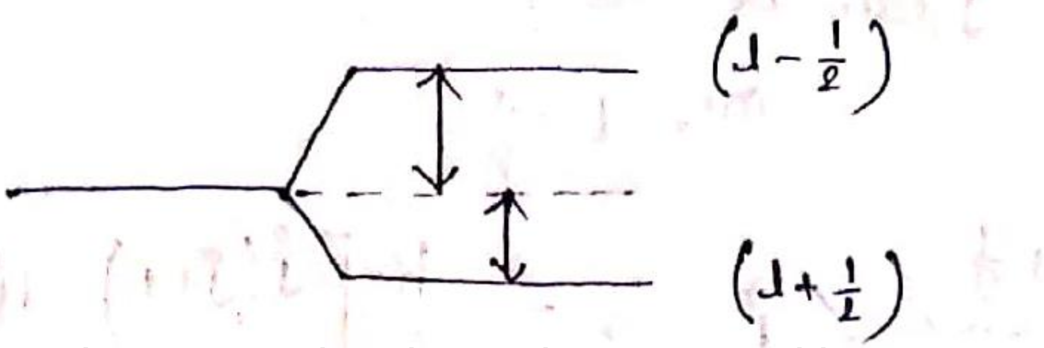
Consider the case for $j = l - 1/2$

$$\langle \vec{l} \cdot \vec{s} \rangle = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$$

$$\langle \vec{l} \cdot \vec{s} \rangle = \frac{\hbar^2}{2}\left[(l - \frac{1}{2})\left(l + \frac{1}{2}\right) - l(l+1) - \frac{3}{4}\right]$$

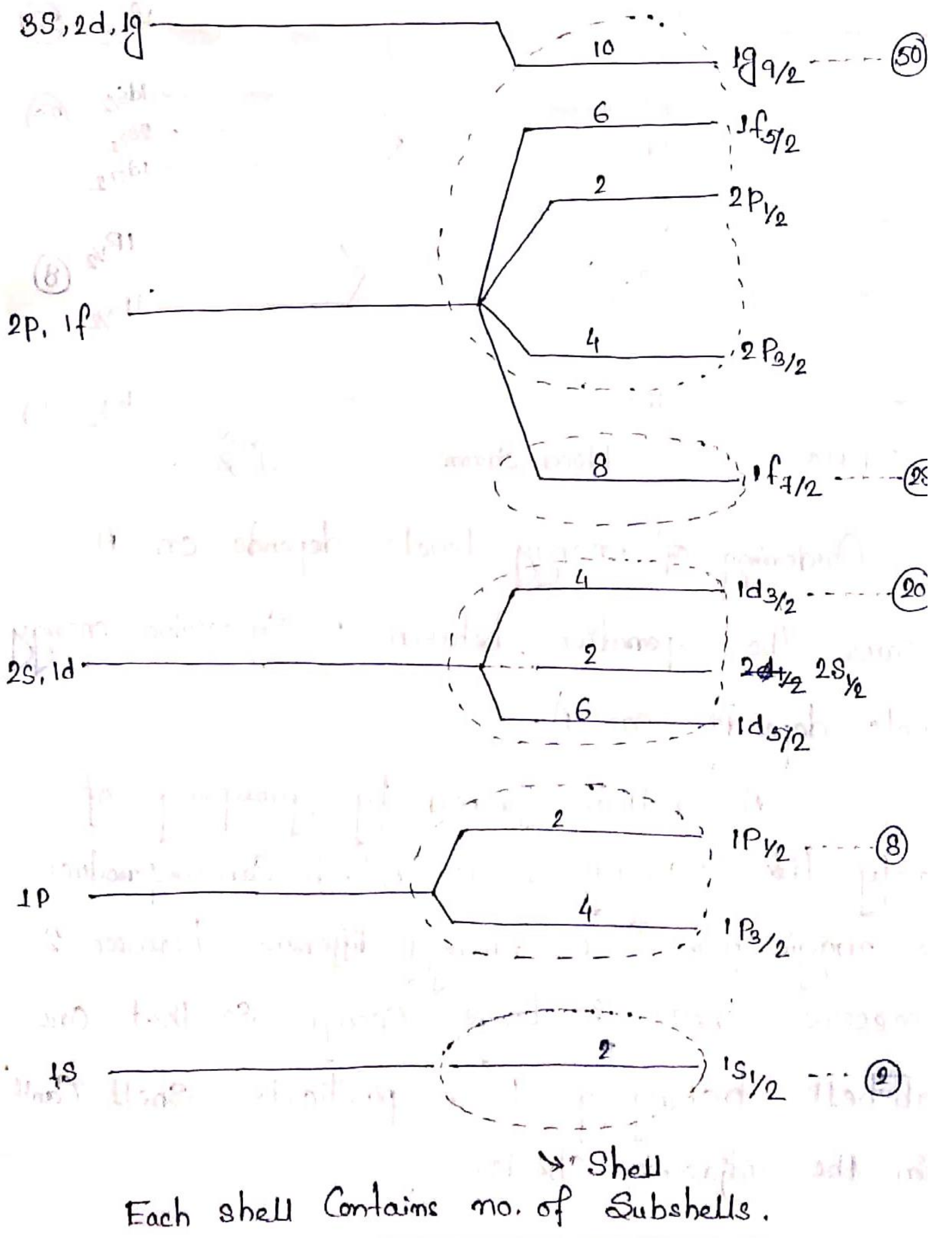
$$\langle \vec{l} \cdot \vec{s} \rangle = -\frac{(l+1)\hbar^2}{2}$$

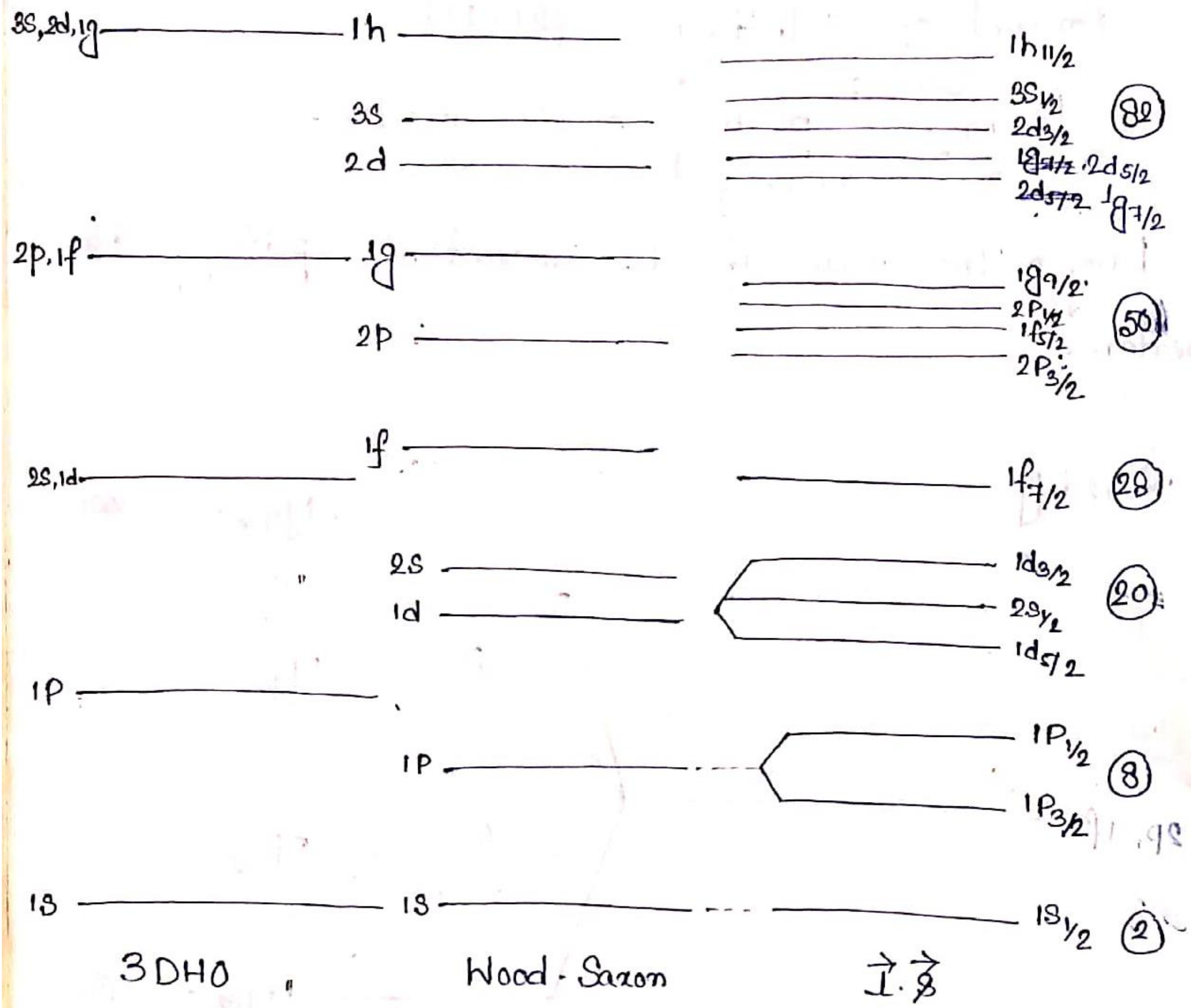
- $\langle \vec{l} \cdot \vec{s} \rangle_{l+\frac{1}{2}} = +\frac{l\hbar^2}{2}$ $\langle \vec{l} \cdot \vec{s} \rangle_{l-\frac{1}{2}} = -\frac{(l+1)\hbar^2}{2}$
- Each energy level splits into two sub-levels.
- It is seen that the energy level corresponding to $j = l - 1/2$ lies above that corresponding to $j = l + 1/2$ level with respect to the unsplitted level as $f(r) < 0$.
- The amount of splitting for the two states would be asymmetric.



- In nuclear physics $f(r)$ should be chosen large enough so that energy levels can be separated by an amount of energy of the order of MeV.

- The separation of energy between the splitted levels is $-\frac{(2l+1)\hbar^2}{2}$.
- Larger the value of l, greater is the amount of splitting.
- Each shell contains a number of subshells.
- A shell is formed by grouping of closely lying energy levels to reproduce the magic numbers.
- Subshells within a particular shell can interchange their position within that shell only, they can not leave their shell to join the adjacent shell.
- The energy difference between two successive energy levels is large enough so that any subshell belonging to a shell can not join the adjacent one.
- Ordering of energy levels depends on A because the separation of successive energy levels depends on A.





3S, 2d, 1g
2p, 1f
2S, 1d
1P
1S
1h
3S
2d
1g
2P
1f
2S
1d
1P
1S
1h11/2
3S1/2
2d3/2
2d5/2
1g7/2
1g9/2
2P1/2
1f5/2
2P3/2
1f7/2
1d3/2
2S1/2
1d5/2
1P1/2
1P3/2
1S1/2
82
50
28
20
8
2
3DHO
Wood - Saxon

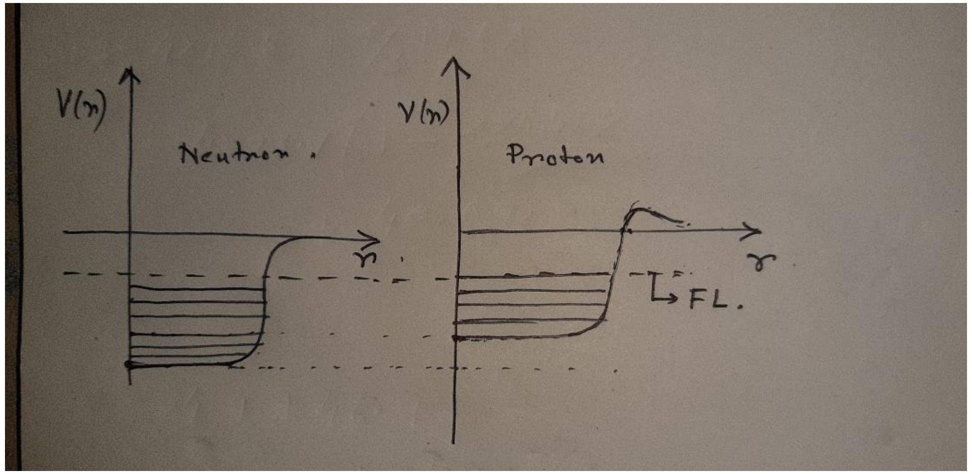
- Upto $N/Z = 28$ the shell structure is independent of the choice of form of the potential.
- When A increases cross-over between the adjacent energy levels can occur. But it must be restricted to the particular shell only. No energy level should leave its own shell. E.g., $1g_{\frac{9}{2}}$ can go below $2p_{\frac{1}{2}}$ depending upon the form of the potential or A.
- In order to make agreement between predicted value and experimental value of ground state spin one must choose the form of potential correctly for heavy nuclei.

  If $A > 50$ $1d_{\frac{3}{2}}$ and $2s_{\frac{1}{2}}$ can be flipped.

  If $A > 70$ $2p_{\frac{1}{2}}$ and $1f_{\frac{5}{2}}$ can be flipped.
- The last filled up energy level of a particular nucleon in a nucleus is known as nuclear Fermi energy level which remains almost the same for all the nuclei. (variation is about 5 MeV).

  The bottom energy level being fixed, the energy gap between highest energy level and the bottom energy level is practically constant. Hence the number of energy levels would be grater for large A, consequently, the energy gap between two successive energy levels would decrease.

$$V(r) = \frac{Ze^2}{4\pi\epsilon_0 R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right]$$



- Due to presence of Coulomb force Wood-Saxon potential is different for neutrons and protons. However, for lighter nuclei one can neglect this effect.
- The potential will be lifted up for proton at $r = 0$ and $r = R$ compared to neutron but the amount of lifting is greater at $r = 0$ compared to $r = R$.
- Energy upto a certain level for both neutrons protons should be the same. Otherwise the inter-conversion of neutron-proton would occur resulting in formation of unstable nuclei.
- The difference between bottom and top of neutron energy levels being greater compared to that for protons, more energy levels can be accommodated between them.
- As the number of energy levels for neutrons is greater compared to that for protons, therefore, the number of neutrons filling up energy levels of a particular nucleus will be greater compared to the number of protons. Hence, for mid-weight and heavy-weight nuclei we observe $N > Z$.
- As A increases the average separation between successive energy levels decreases hence it is inversely proportional to A as observed in Asymmetry term in Liquid Drop Model.

Applications:

1. Spin and Parity of nuclei at ground state:

Spin of a nucleus is the total angular momentum of the corresponding nucleus.

Spin of nucleus = Total angular momentum $= J^{\pi}$

Parity $\pi = (-1)^{l}$

Case I ($A = even$; e-e nuclei)

Nucleons will be paired up in maximum possible ways.

Each pair contributes zero angular momentum, hence total angular momentum for e-e nuclei should be equal to zero.

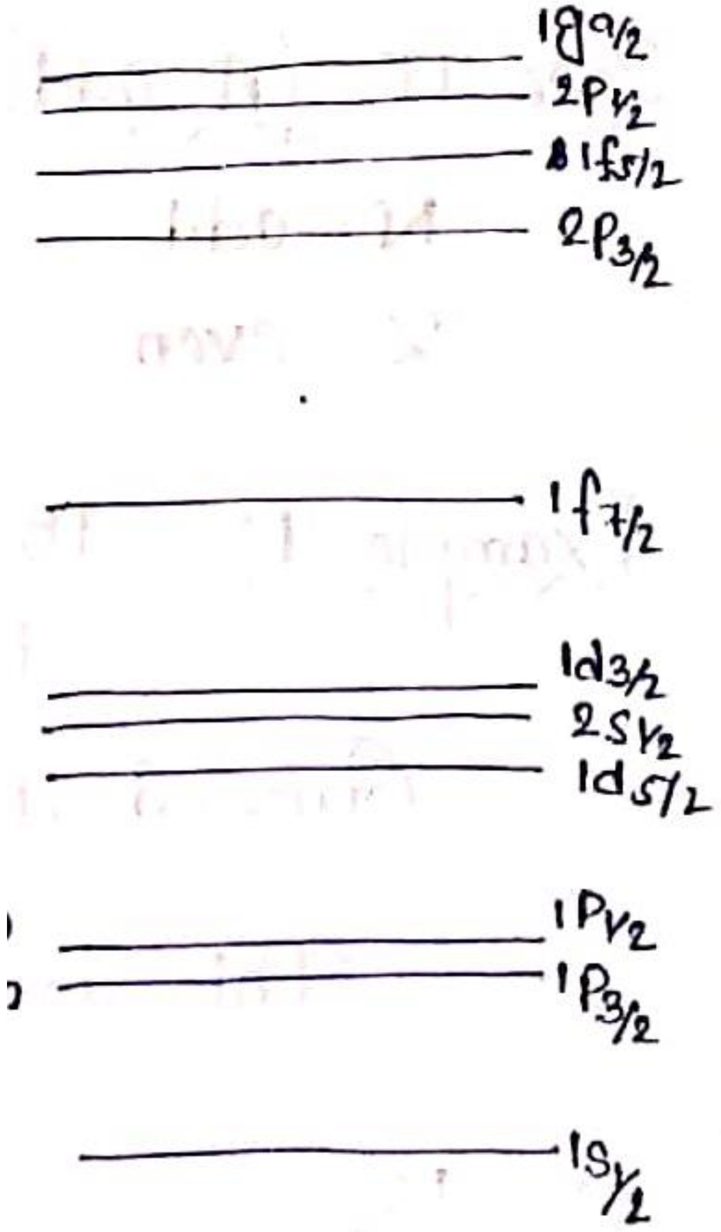
Pairing lowers the energy of the nucleus.

$$\vec{j_1} + \vec{j_2} = \vec{j}$$

$$J^{\pi} = 0^{+}$$

Parity is even for all e-e nuclei as pairing occurs for a particular state, Hence both the nucleons have same l value (odd or even).

This prediction is more or less consistent with the experimental results.



Case II ($A = odd$)

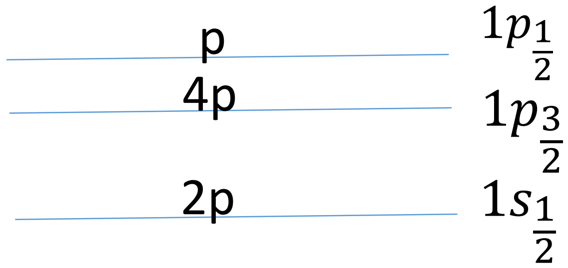
$N = odd \qquad Z = even$

$N = even \qquad Z = odd$

Examples:

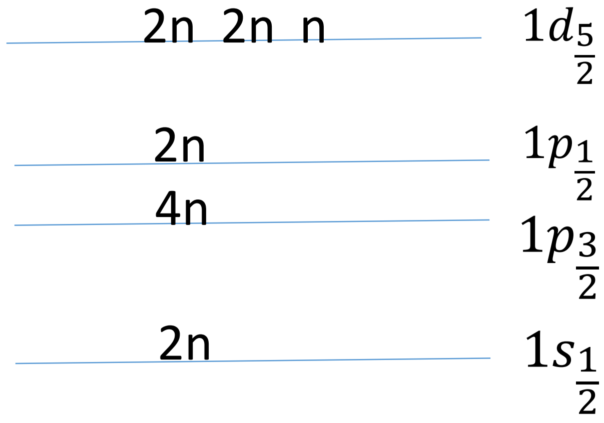
1. ${}^{15}_{7}N(Z = 7, N = 8)$

$Z = 7; \;\; J^{\pi} = \frac{1}{2}^{-}$



2. ${}^{27}_{14}Si(Z = 14, N = 13)$

$N = 13; \;\; J^{\pi} = \frac{5}{2}^{+}$



3. $^{61}_{28}Ni(Z = 28, N = 33)$

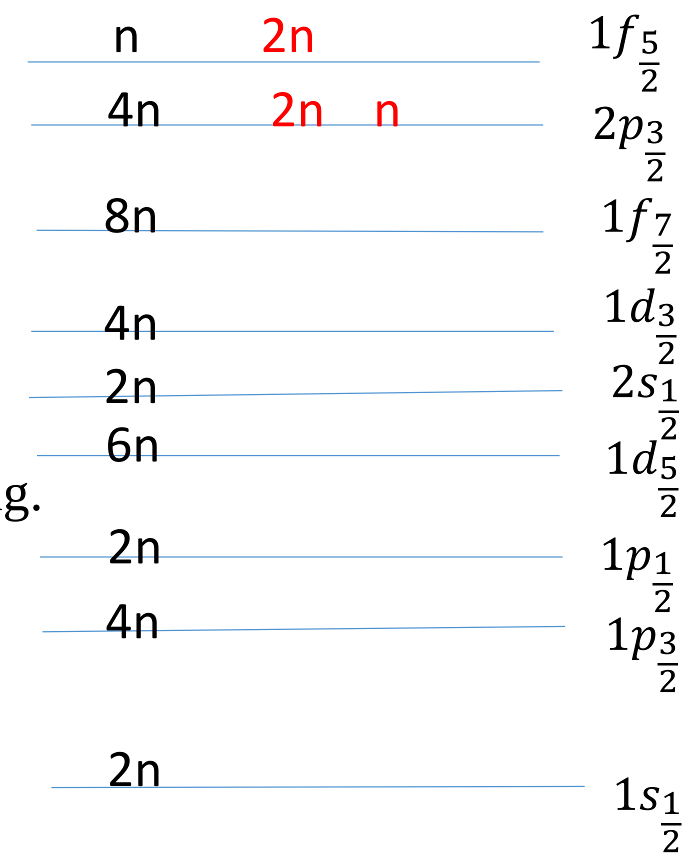
Prediction: $\frac{5}{2}^{-}$ (for $f, l = 3$)   Experiment: $\frac{3}{2}^{-}$ (for $p, l = 1$)

Making of a pair lowers energy and amount of decrement in energy depends on l − value of the state. Hence decrease in energy of f state due to pairing is greater compared to that in p state.

The energy difference between $1f_{\frac{5}{2}}$ and $2p_{\frac{3}{2}}$ is less compared to the difference between decrement in energy at f and p level due to pairing.

Therefore, $1f_{\frac{5}{2}}$ will be filled up by neutrons before the $2p_{\frac{3}{2}}$ state.

Hence, $2p_{\frac{3}{2}}$ state will contain the unpaired neutron. As a result the observed ground state spin parity would be $\frac{3}{2}^{-}$.



- According to the Extreme Single Particle Shell Model the last unpaired nucleon determines the physical properties of a nucleus.
- Predict the ground state spin-parity : $^{17}_{8}O$ & $^{15}_{8}O$.
- Odd-odd nuclei cases will be discussed later on.

Quadrupole moment of nuclei in ground state

$$Q_{sp} = -\frac{2j-1}{2j+2}\langle r^2\rangle \qquad\qquad Q_{sn} = \frac{Z}{A^2}Q_{sp}$$

- $\langle r^2\rangle$ is the mean square radius of the charge distribution. $\langle r^2\rangle = \frac{3}{5}R^2 = \frac{3}{5}R_0^2A^{\frac{2}{3}}$
- Q_{sp} and Q_{sn} denote quadrupole moments for unpaired proton nucleus & unpaired neutron nucleus respectively.
- Q_{sn} is much smaller compared to Q_{sp} which is obvious from their expressions.
- Depending upon value of j, Q can be greater or less than zero.
- If $Q = 0$, the nucleus is spherical.
- For $Q > 0$ it is prolate spheroid and for $Q < 0$ it is oblate spheroid.
- Predicted values of Q are much smaller compared to the experimental values for many nuclei although the signatures are found to be correct.
- Shell Model can not account for these discrepancies. Theses nuclei see to acquire permanent deformation due to rotational motion of the nucleus.
- Estimate the quadrupole moment of ${}^{41}_{19}K$.

NUCLEAR MODELS(L3)

Odd-odd nuclei:(A=even) (4)

N=odd, Z= odd

Brennan-Berstein Rules: $B = j_p - l_p + j_n - l_n$

Rule-1: If $B = 0; j = |j_p - j_n|$; $\pi = -ve$

Rule-2: If $B = \pm 1 ; j = |j_p \pm j_n|$; $\pi = \pm ve$ $(N = Z)$

Rule-3: If $B = \pm 1 ; j = |j_p + j_n - 1|$; $\pi = +ve$ $(N \neq Z)$

Example: ${}^{56}_{27}Co(Z = 27, N = 29)$

$B = j_p - l_p + j_n - l_n = \frac{7}{2} - 3 + \frac{3}{2} - 1 = +1$

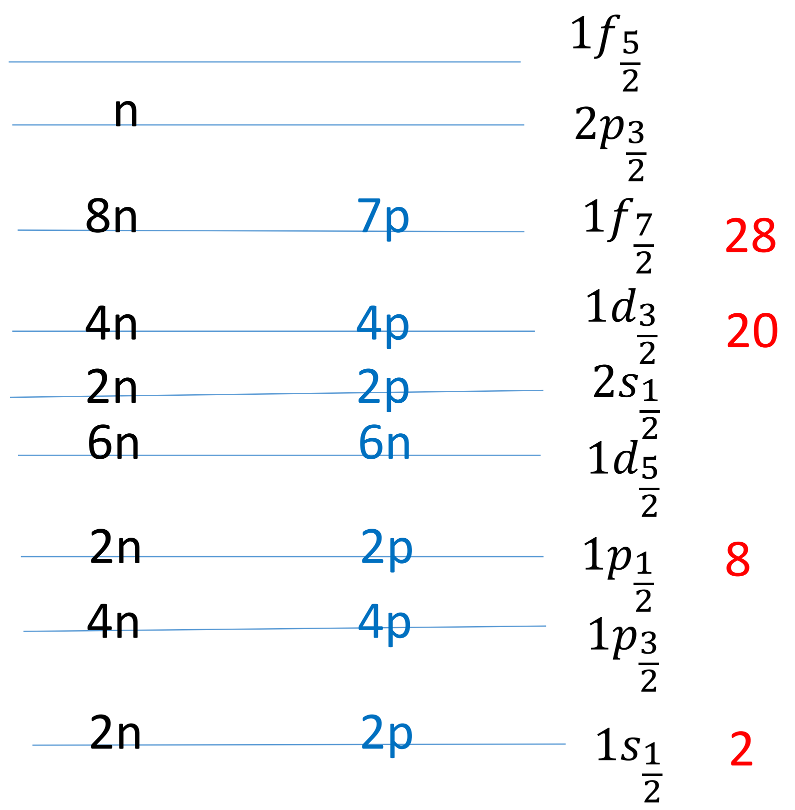
$j = |j_p + j_n - 1| = 4$; $\pi = (-1)^3 \times (-1)^1 = +1$

Predict the ground state spin-parity of the following nuclei:

i) ${}^{42}_{19}K(Z = 19, N = 23)$

ii) ${}^{80}_{35}Br(Z = 35, N = 45)$

iii) ${}^{26}_{13}Al(Z = 13, N = 13)$



Prediction of excites states of the nuclei

Example: ${}^{17}_{8}O(Z = 8, N = 9)$

Ground state : $J^{\pi} = \frac{5}{2}^{+}$; 1st excited state: $J^{\pi} = \frac{1}{2}^{+}$ (unpaired n at $1d_{\frac{5}{2}}$ goes to $2s_{\frac{1}{2}}$)

2nd excited state: $J^{\pi} = \frac{1}{2}^{-}$; Breaking of a pair needs (≈2)MeV energy. Excited state must lie above 2 MeV. Pairing is made at d state and unpaired n stays at p state.

3rd excited state: $J^{\pi} = \frac{5}{2}^{-}$

$n \rightarrow 1p_{\frac{1}{2}} \rightarrow \frac{1}{2}^{-}$; $j_1 = \frac{1}{2}$

$n \rightarrow 1d_{\frac{5}{2}} \rightarrow \frac{5}{2}^{+}$; $j_2 = \frac{5}{2}$

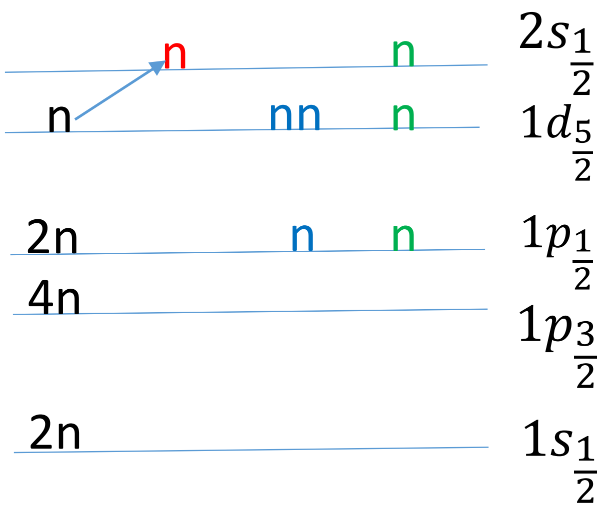
$n \rightarrow 1s_{\frac{1}{2}} \rightarrow \frac{1}{2}^{+}$; $j_3 = \frac{1}{2}$

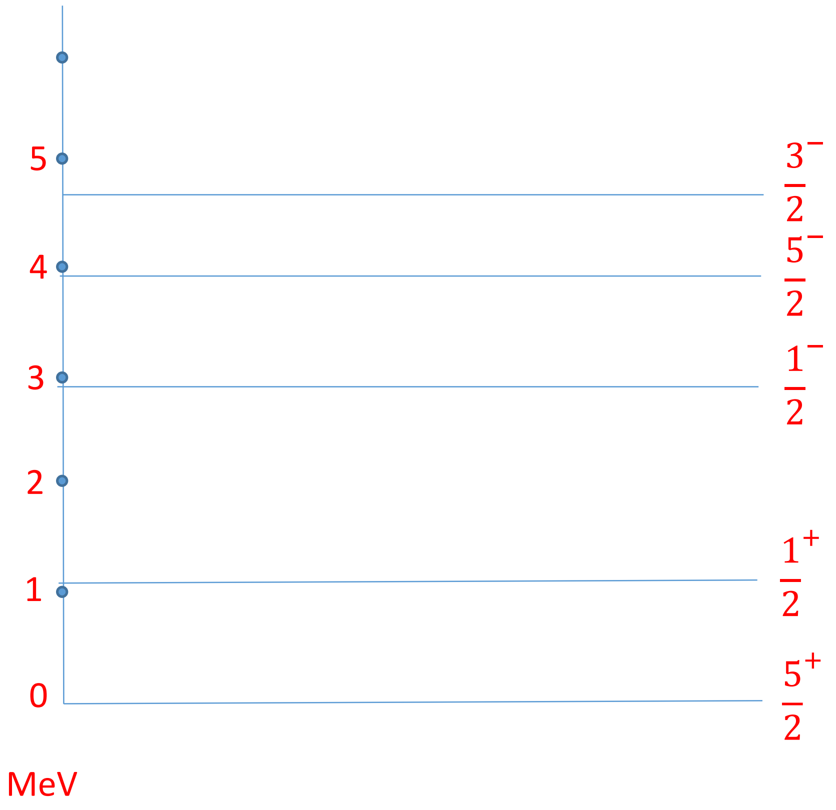
$j_1 + j_3 = 1,0$

There is a chance of obtaining

$j = \frac{5}{2}$ by adding 0 with $j_2 = \frac{5}{2}$

Parity is $(-) \times (+) \times (+) = -$





Example: ${}^{41}_{20}Ca(Z = 20, N = 21)$

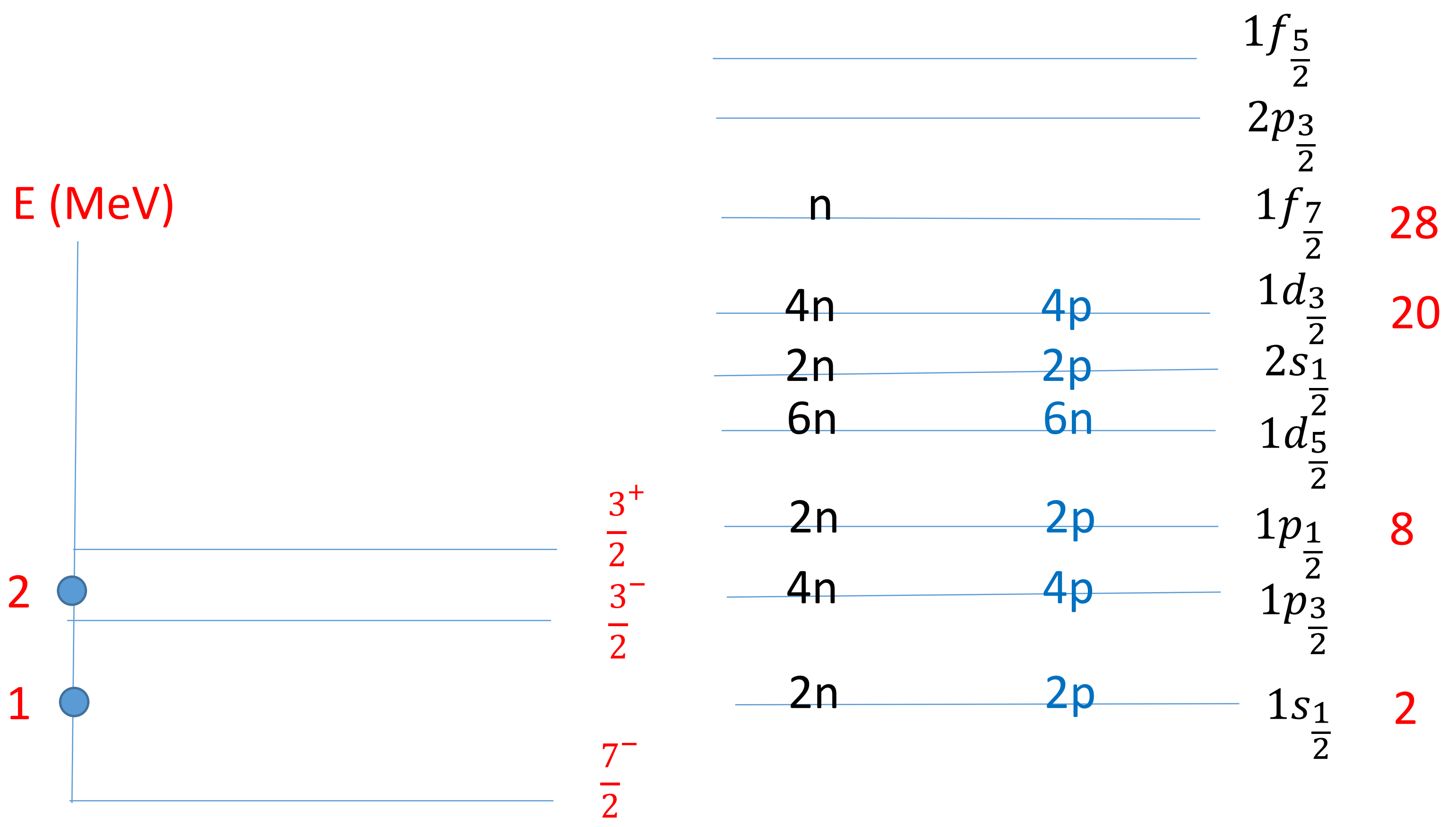
Last unpaired neutron stays at $1f_{\frac{7}{2}}$

Hence ground state spin-parity is $J^{\pi} = \frac{7}{2}^{-}$

$1f_{\frac{7}{2}} \rightarrow 2p_{\frac{3}{2}}$ giving rise to 1st excited state at $J^{\pi} = \frac{3}{2}^{-}$

$1d_{\frac{3}{2}} \rightarrow 1f_{\frac{7}{2}}$ giving rise to 2nd excited state at $J^{\pi} = \frac{3}{2}^{+}$

Observe the separation between 1st and 2nd excited states.



Example: ${}^{41}_{21}Sr (Z = 21, N = 20)$

Last unpaired proton stays at $1f_{\frac{7}{2}}$

Hence ground state spin-parity

is $J^{\pi} = \frac{7}{2}^{-}$

$1f_{\frac{7}{2}} \rightarrow 2p_{\frac{3}{2}}$ giving rise to 1st

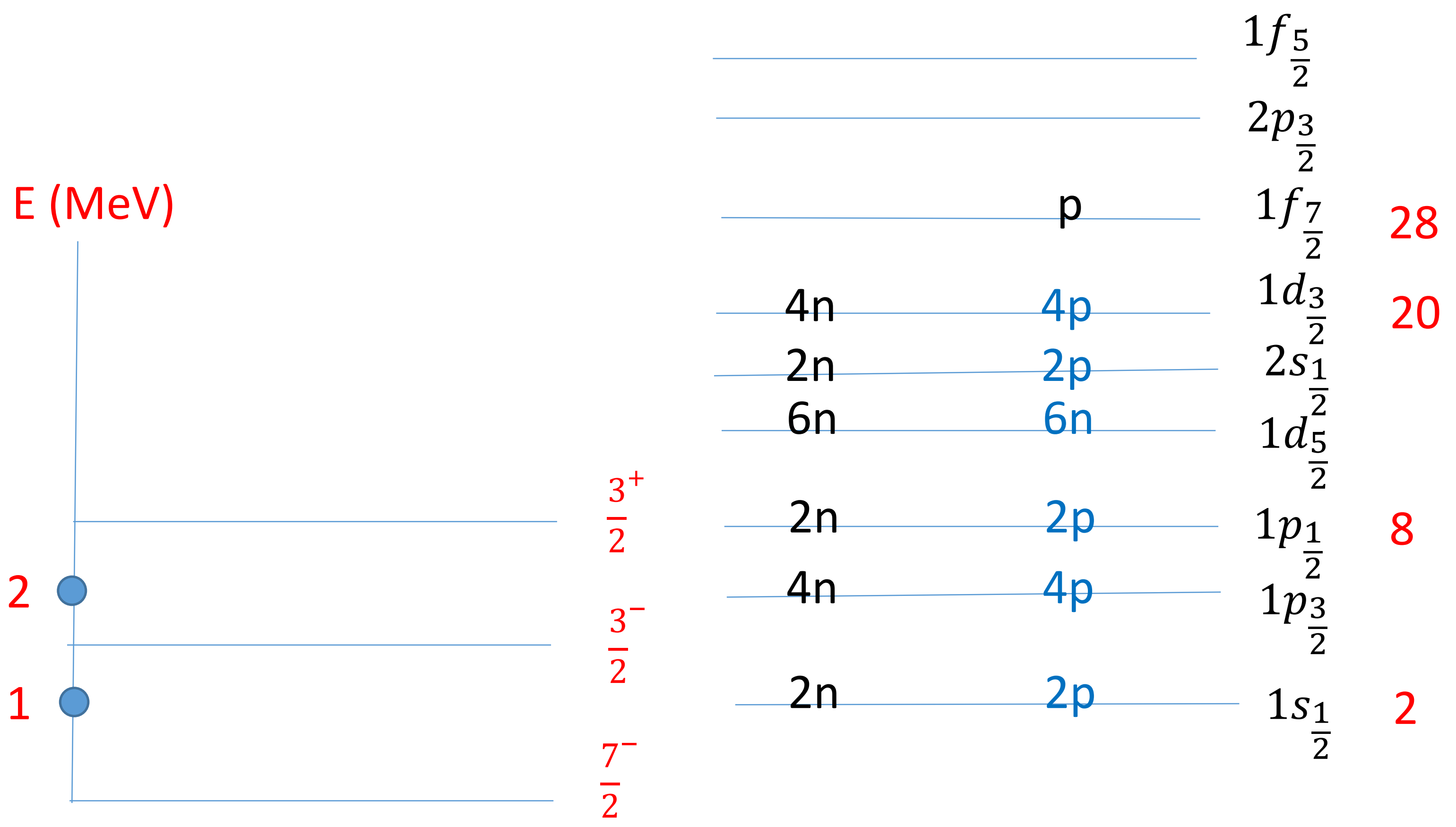
excited state at $J^{\pi} = \frac{3}{2}^{-}$

$1d_{\frac{3}{2}} \rightarrow 1f_{\frac{7}{2}}$ giving rise to 2nd

excited state at $J^{\pi} = \frac{3}{2}^{+}$

Observe the separation between 1st

and 2nd excited states.



The difference between two excited states in case of ${}^{41}_{20}Ca$ is less compared to ${}^{41}_{21}Sr$. Why?

Reason: Coulomb repulsion acts between protons but does not act between neutrons.

Magnetic dipole moment of nuclei in ground state

- Direction of applied magnetic field is chosen along $Z -$ direction.
- Expectation value of magnetic moment is measured along the direction of applied magnetic field.

Case-I (e-e nuclei):

All the nucleons are paired up and, each pair contributes zero angular momentum, hence, $\langle \mu_Z \rangle = 0$

Case-II ($A = odd$):

- $N = odd$; $Z = even$
- $N = even$; $Z = odd$
- Only the unpaired nucleon will contribute to the magnetic dipole moment of the corresponding nucleus.
- $\mu_Z = (g_l l_Z + g_s s_Z)\frac{\mu_N}{\hbar}$ $\quad where \ Nuclear \ magneton, \mu_N = \frac{e\hbar}{2m_p}$
- For proton, $g_l = 1 \ and \ g_s = 5.5857$
- For neutron, $g_l = 0 \ and \ g_s = -3.826$
- $\vec{j} = \vec{l} + \vec{s}$ and $j_Z = l_Z + s_Z$, different combinations of l_Z and s_Z can give the same j_Z
- Following addition of angular momentum l, s, j, j_Z will form definite eigenstates.

Continued…..

- Since $s_Z = \pm\frac{1}{2}$; $j = l \pm \frac{1}{2}$

$j = l + \frac{1}{2}$

$$\langle \mu_Z \rangle = \langle j, m_j = j | \mu_Z | j, m_j = j \rangle$$

$$= \left\langle m_l = l, m_s = \frac{1}{2} \middle| \mu_Z \middle| m_l = l, m_s = \frac{1}{2} \right\rangle$$

$$= \frac{\mu_N}{\hbar} \left\langle m_l = l, m_s = \frac{1}{2} \middle| g_l l_Z + g_s s_Z \middle| m_l = l, m_s = \frac{1}{2} \right\rangle$$

$$= \frac{\mu_N}{\hbar} \left[l\hbar g_l + \frac{1}{2}\hbar g_s \right]$$

$$= \mu_N [l + 2.7929]$$

$$= \mu_N [j + 2.2929]$$

- $m_j = m_l + m_s \Rightarrow l + \frac{1}{2} = (l, l-1, l-2, \ldots, -l) + \left(+\frac{1}{2}, -\frac{1}{2}\right) = \left(l + \frac{1}{2}\right); \left(l - \frac{1}{2}\right); \left(l - \frac{1}{2}\right); (l - \frac{3}{2}) \ldots..$
- Only the 1st combination leads to $m_j = l + \frac{1}{2}$
- For proton : $(\mu_Z)_{proton} = \mu_N [j + 2.2929]$
- For neutron : $(\mu_Z)_{neutron} = -1.913 \mu_N$

Continued….

$j = l - \frac{1}{2}$

- $m_j = m_l + m_s \Rightarrow l - \frac{1}{2} = (l, l-1, l-2, \ldots, -l) + \left(+\frac{1}{2}, -\frac{1}{2}\right) = \left(l + \frac{1}{2}\right); \left(l - \frac{1}{2}\right); \left(l - \frac{1}{2}\right); (l - \frac{3}{2}) \ldots..$
- The state $m_j = l - \frac{1}{2}$; i.e., $\left|j, m_j = j = l - \frac{1}{2}\right\rangle = a\left|m_l = l, m_s = -\frac{1}{2}\right\rangle + b\left|m_l = l-1, m_s = \frac{1}{2}\right\rangle$
- $|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$; where, $|\psi_1\rangle$ and $|\psi_2\rangle$ are definite eigenstates of l_Z and s_Z.

$$\langle\mu_Z\rangle = \langle\psi|\mu_Z|\psi\rangle = |a|^2\langle\psi_1|\mu_Z|\psi_1\rangle + |b|^2\langle\psi_2|\mu_Z|\psi_2\rangle + a^*b\langle\psi_1|\mu_Z|\psi_2\rangle + ab^*\langle\psi_2|\mu_Z|\psi_1\rangle$$

$$= |a|^2\langle\psi_1|\mu_Z|\psi_1\rangle + |b|^2\langle\psi_2|\mu_Z|\psi_2\rangle \quad \text{------------} \quad (1)$$

$$\langle\psi_1|\mu_Z|\psi_1\rangle = \left\langle m_l = l, m_s = -\frac{1}{2}\right|\mu_Z\left|m_l = l, m_s = -\frac{1}{2}\right\rangle$$

$$= \frac{\mu_N}{\hbar}\left\langle m_l = l, m_s = -\frac{1}{2}\right|g_l l_Z + g_s s_Z\left|m_l = l, m_s = -\frac{1}{2}\right\rangle$$

$$= \frac{\mu_N}{\hbar}\left[l\hbar g_l - \frac{1}{2}\hbar g_s\right] = \mu_N\left[g_l l - \frac{1}{2}g_s\right]$$

Continued….

$$\langle\psi_2|\mu_Z|\psi_2\rangle = \left\langle m_l = l-1, m_s = \frac{1}{2}\middle|\mu_Z\middle|m_l = l-1, m_s = \frac{1}{2}\right\rangle$$

$$= \frac{\mu_N}{\hbar}\left\langle m_l = l-1, m_s = \frac{1}{2}\middle|g_l l_Z + g_s s_Z\middle|m_l = l-1, m_s = \frac{1}{2}\right\rangle$$

$$= \frac{\mu_N}{\hbar}\left[(l-1)\hbar g_l + \frac{1}{2}\hbar g_s\right] = \mu_N\left[g_l(l-1) + \frac{1}{2}g_s\right]$$

Determination of a & b:

$$J_\pm\left|j, m_j\right\rangle = \hbar\sqrt{(j \mp m_j)(j \pm m_j + 1)}\left|j, m_j \pm 1\right\rangle \quad where, J_\pm = J_x \pm i\,J_y$$

$J_+\left|j, m_j = j\right\rangle = 0$ ----------- (2)

$$J_+|\psi_1\rangle = (l_+ + s_+)\left|m_l = l, m_s = -\frac{1}{2}\right\rangle = \hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\left|m_l = l, m_s = \frac{1}{2}\right\rangle = \hbar\left|m_l = l, m_s = \frac{1}{2}\right\rangle$$

$$J_+|\psi_2\rangle = (l_+ + s_+)\left|m_l = l-1, m_s = \frac{1}{2}\right\rangle = \hbar\sqrt{(l-l+1)(l+l-1+1)}\left|m_l = l, m_s = \frac{1}{2}\right\rangle = \hbar\sqrt{2l}\left|m_l = l, m_s = \frac{1}{2}\right\rangle$$

From Eq. (2) we obtain $\left(\mathrm{a} + \mathrm{b}\sqrt{2l}\right)\hbar\left|m_l = l, m_s = \frac{1}{2}\right\rangle = 0 \Rightarrow a = -b\sqrt{2l}$

Now, $\langle\psi|\psi\rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1 \Rightarrow |b|^2 = \frac{1}{1+2l}$ & $|a|^2 = \frac{2l}{1+2l}$

From Eq. (1)

$$\langle\mu_Z\rangle = |a|^2\langle\psi_1|\mu_Z|\psi_1\rangle + |b|^2\langle\psi_2|\mu_Z|\psi_2\rangle$$

$$= \frac{2l}{2l+1}\left[g_l l - \frac{g_s}{2}\right]\mu_N + \frac{1}{2l+1}\left[g_l(l-1) + \frac{g_s}{2}\right]\mu_N$$

For $j = l - \frac{1}{2}\ or\ l = j + \frac{1}{2}$

$$\langle\mu_Z\rangle = \frac{\mu_N}{2l+1}\left[g_l(2l^2 + l - 1) - \frac{g_s}{2}(2l-1)\right]$$

$$= \frac{\mu_N}{2\left(j+\frac{1}{2}\right)+1}\left[g_l\left(2\left(j+\frac{1}{2}\right)^2 + \left(j+\frac{1}{2}\right) - 1\right) - g_s\left(2\left(j+\frac{1}{2}\right) - 1\right)\right]$$

$$= \frac{\mu_N j}{2j+2}\left[g_l(2j+3) - g_s\right]$$

For proton $g_l = 1 \;\; and \; g_s = 5.5857$

$$\langle \mu_Z \rangle = \frac{\mu_N j}{2j+2}[(2j+3) - 5.5857]$$

$$\langle \mu_Z \rangle_{proton} = \frac{\mu_N j}{j+1}[j - 1.2928]$$

For neutron $g_l = 0 \;\; and \; g_s = -3.826$

$$\langle \mu_Z \rangle = \frac{\mu_N j}{j+1}[-g_s]$$

$$\langle \mu_Z \rangle_{neutron} = \frac{1.913\mu_N j}{j+1}$$

- According to the ESPSM the last unpaired nucleon is responsible for the ground state properties of nuclei.
- Although the predictions of this model is by and large consistent with the experimental observations with respect to spin and parity of the nuclei in ground state but not at par with the observed values of magnetic moments of the nuclei in ground state.

Schmidt Diagram:

Schmidt diagram represents the upper bound and lower bound of magnetic moments of nuclei in their respective ground states.

For neutron:

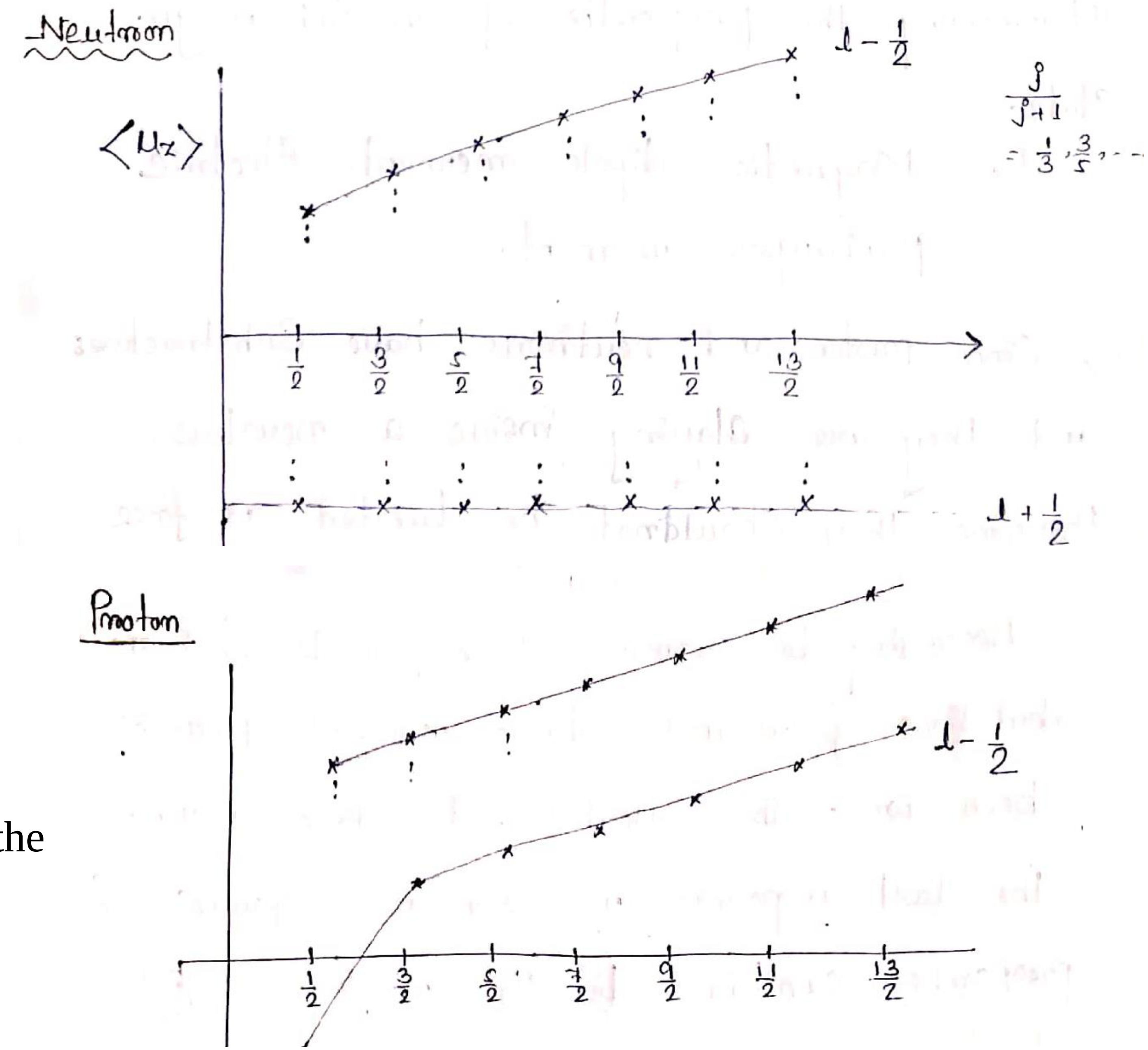
The upper bound corresponds to $j = l - \frac{1}{2}$ and the lower bound is for $j = l + \frac{1}{2}$

For proton:

The upper bound corresponds to $j = l + \frac{1}{2}$ and the lower bound is for $j = l - \frac{1}{2}$

The limits are obtained from the formulae obtained in earlier slides.

The observed values are found to lie between the bounds shown in Schmidt Diagram.



Reason for mismatch between the theoretical values and the observed experimental values:

- Predicted values of electric quadrupole moments of nuclei deviates by order of magnitude if the calculations are made on the basis of ESPSM.
- ESPSM does not work for predicting the magnetic moments of nuclei in ground states. The paired nucleons may take part in determining these properties of nuclei in ground states.
- Since protons and neutrons have substructures and they stay inside the nucleus, hence they should not be treated as free particle. Therefore, g_s should be modified accordingly.

Suppose g_s changes due to change in its surroundings. In fact, if the value of g_s is changed by 60% due to change in surroundings, the estimated values of μ_Z would change and the Schmidt lines will be compressed resulting in good agreement with the experimental values.

Magnetic dipole moment of ${}^{17}_{8}O(N = 9, Z = 8)$ in ground state

The last unpaired neutron lies at $1d_{\frac{5}{2}}$

Ground state spin-parity : $J^{\pi} = \frac{5}{2}^{+}$; $l = 2$

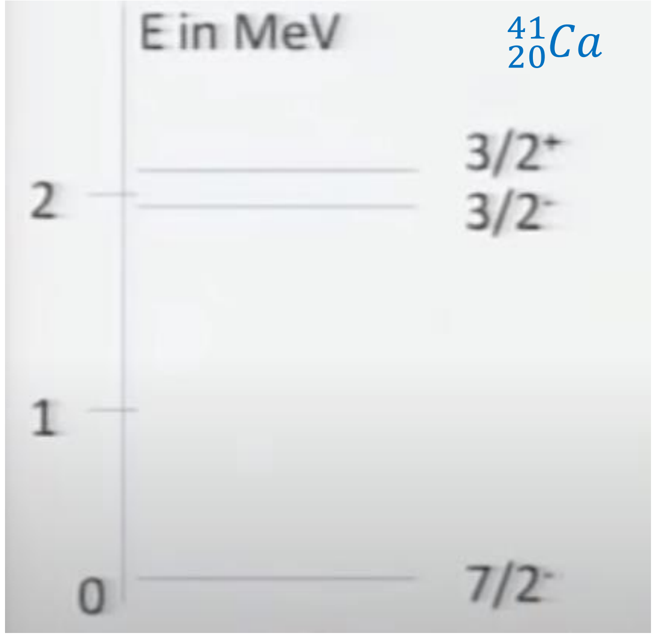
We have to use $j = l + \frac{1}{2}$ formula for neutron for estimating μ_Z.

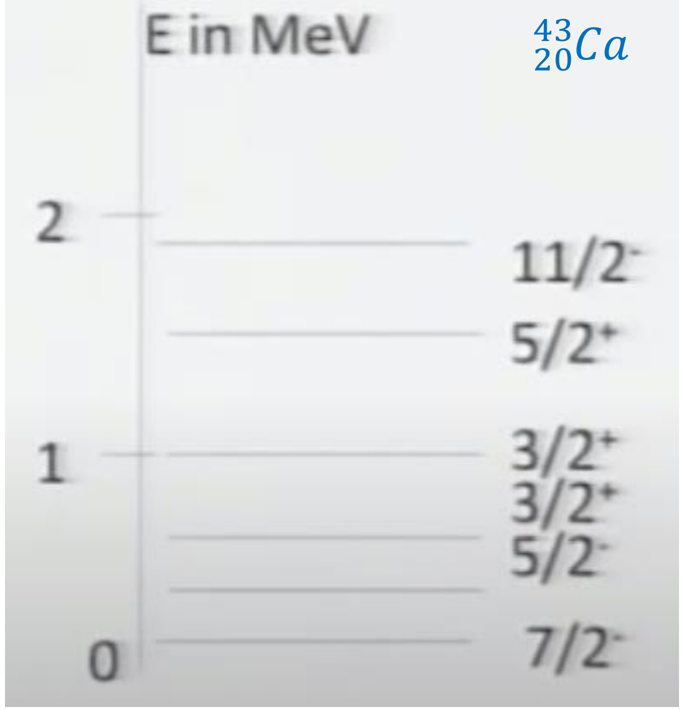
The value is $-1.913\mu_N$ (constant)

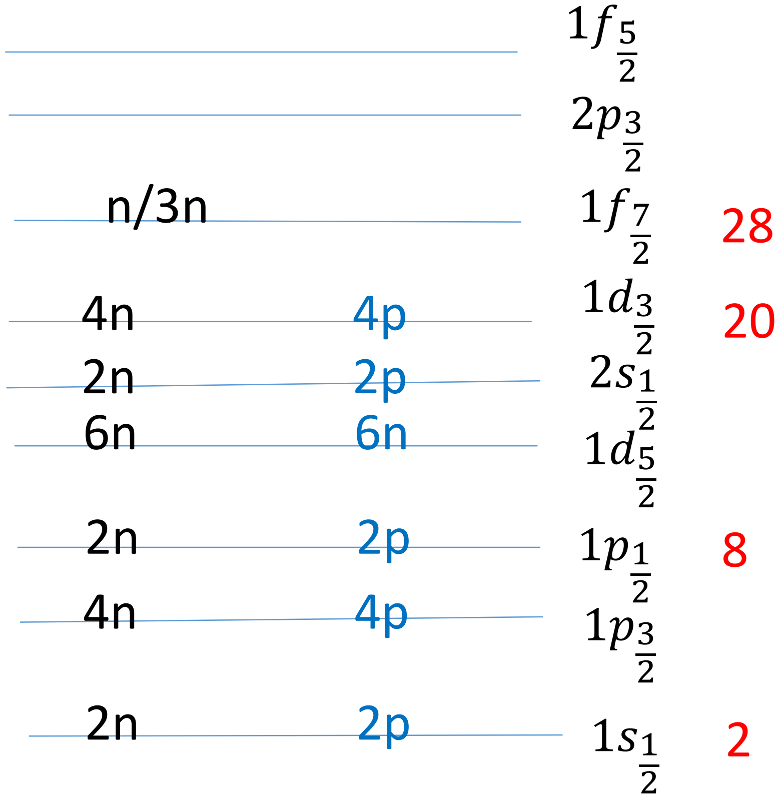
NUCLEAR MODELS(L4)

Example: ${}^{41}_{20}Ca(Z = 20, N = 21)$; ${}^{43}_{20}Ca(Z = 20, N = 23)$

Last unpaired neutron stays at $1f_{\frac{7}{2}}$







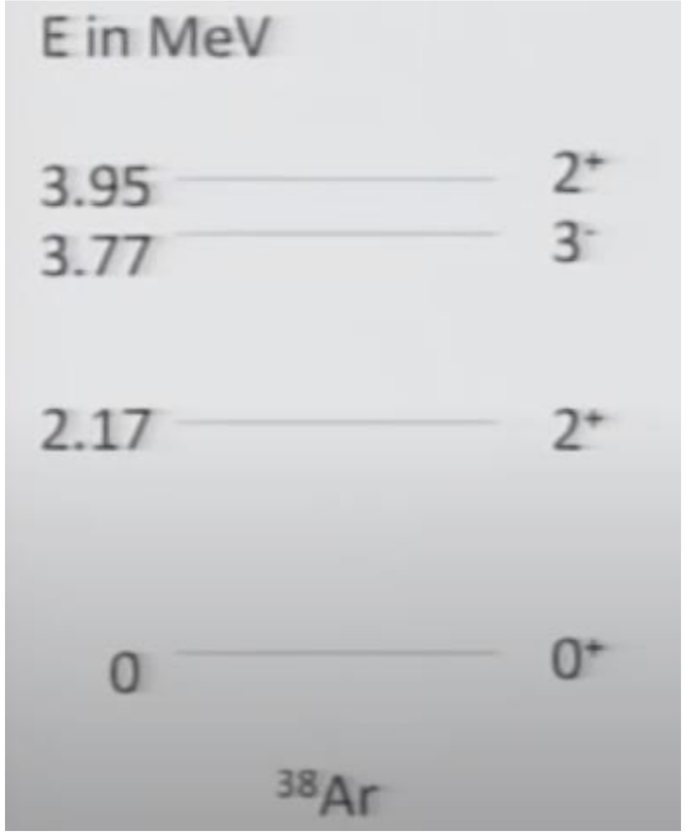
- According to ESPSM the energy states of ${}^{41}_{20}Ca$ & ${}^{43}_{20}Ca$ should have been identical as the last unpaired neutron stays at $1f_{\frac{7}{2}}$ in both the cases.
- To obtain $\frac{3}{2}^+$ state a pair is to be broken which needs $\approx 2\ MeV$ energy, but $\frac{3}{2}^+$ lies near $1\ MeV$ for ${}^{43}_{20}Ca$. Moreover, other excited energy states are obtained. This feature indicates deviation from ESPSM.
- May be instead of last unpaired nucleon, many more nucleons or even the whole nucleus is participating in determining the properties of nuclei. Collective motion of nucleons need to be considered.

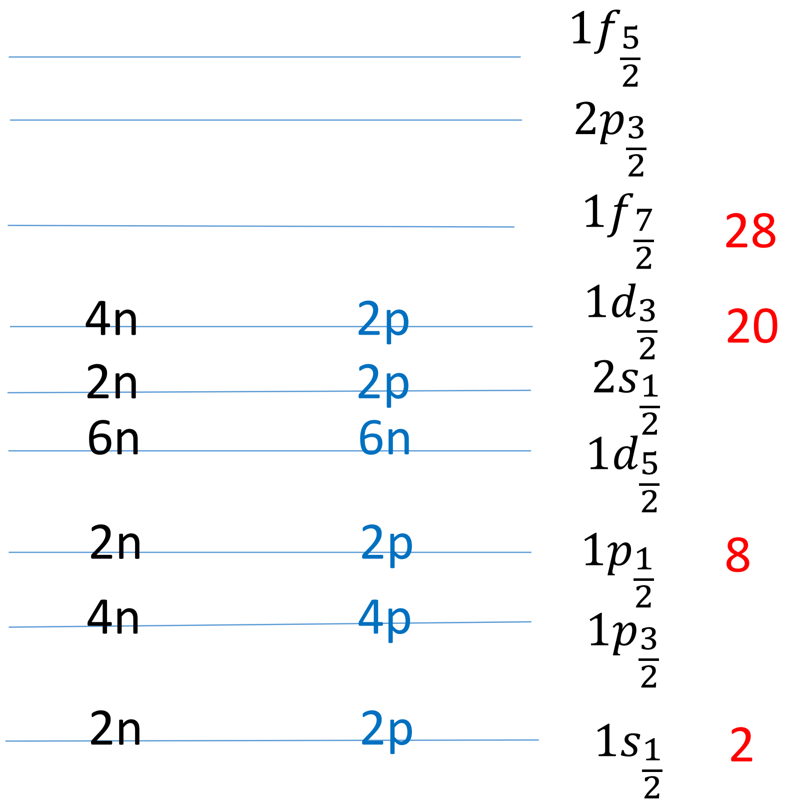
Example: ${}^{38}_{18}Ar(Z = 18, N = 20)$

Last 2 protons stay at $1d_{\frac{3}{2}}$ in pair.

One p is elevated to $1f_{\frac{7}{2}}$ by breaking a pair at $1d_{\frac{3}{2}}$. $j_1 = \frac{7}{2}; j_2 = \frac{3}{2}$

These 2 unpaired p can combine to give $2^-, 3^-, 4^-, 5^-$. Possible explanation for 3^- at 3.77 MeV.





One p is elevated to $1d_{\frac{3}{2}}$ by breaking a pair at $2s_{\frac{1}{2}}$.

$j_1 = \frac{1}{2}; j_2 = \frac{3}{2}$

These 2 unpaired p can combine to give $1^+, 2^+$. But experiment shows this 2^+ at 3.95 MeV. Lower 2^+ state can not be explained. At least 2 MeV energy is needed to break a pair and more energy is needed to move it up. So it does not account for the formation of 2^+ at 2.17 MeV

- ESPSM fails to explain the formation of 1st excited states for e-e nuclei which is 2^+.

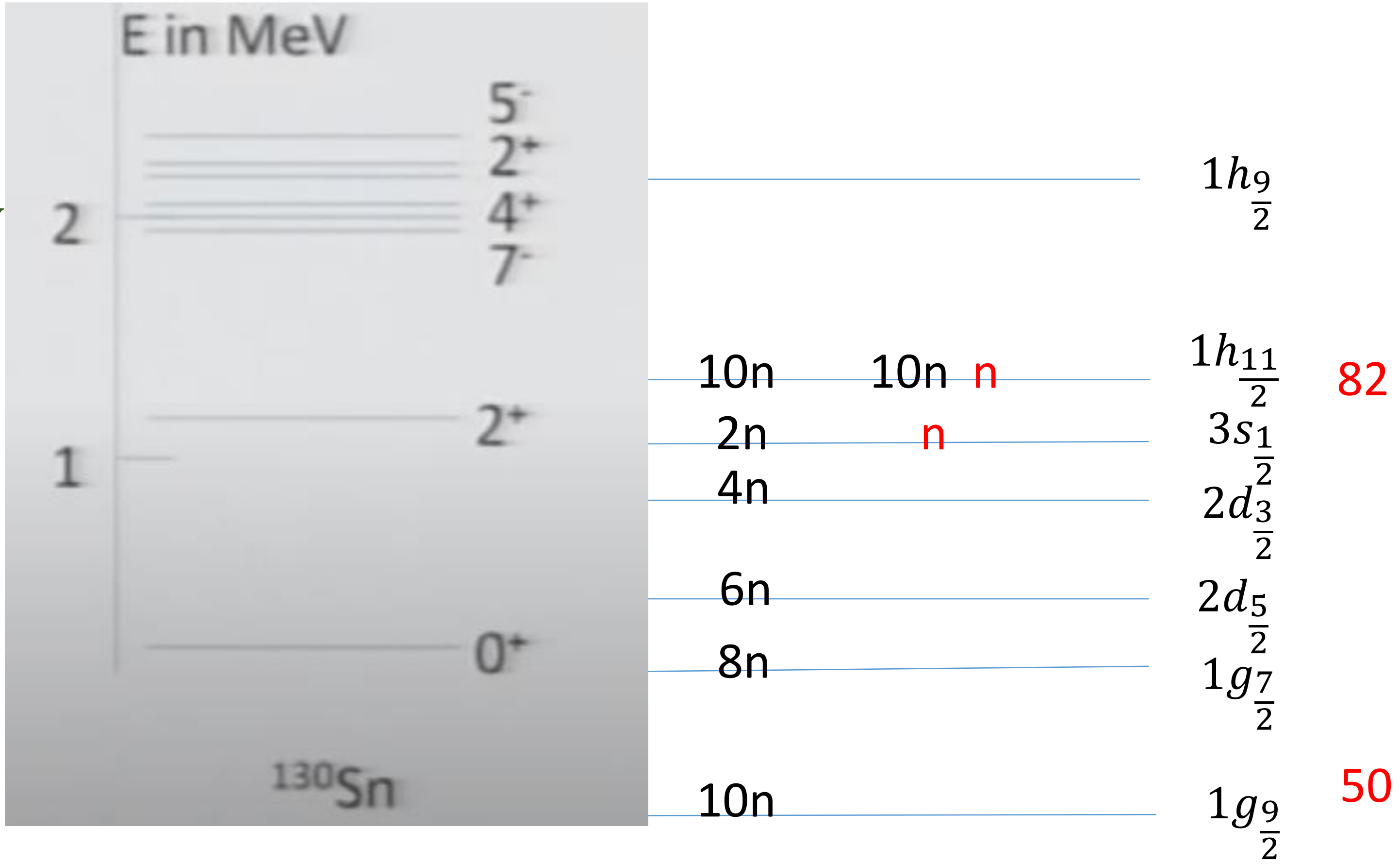
Example: ${}^{130}_{50}Sn(Z = 50, N = 80)$

Last 10 neutrons stay at $1h_{\frac{11}{2}}$ in pairs.

1st excited state 2^+ lies around $1.2\ MeV$

Many other excited states are found to be crowded around $(2 - 2.5)\ MeV$.

Arrangements of neutrons for $N > 50$ are shown.

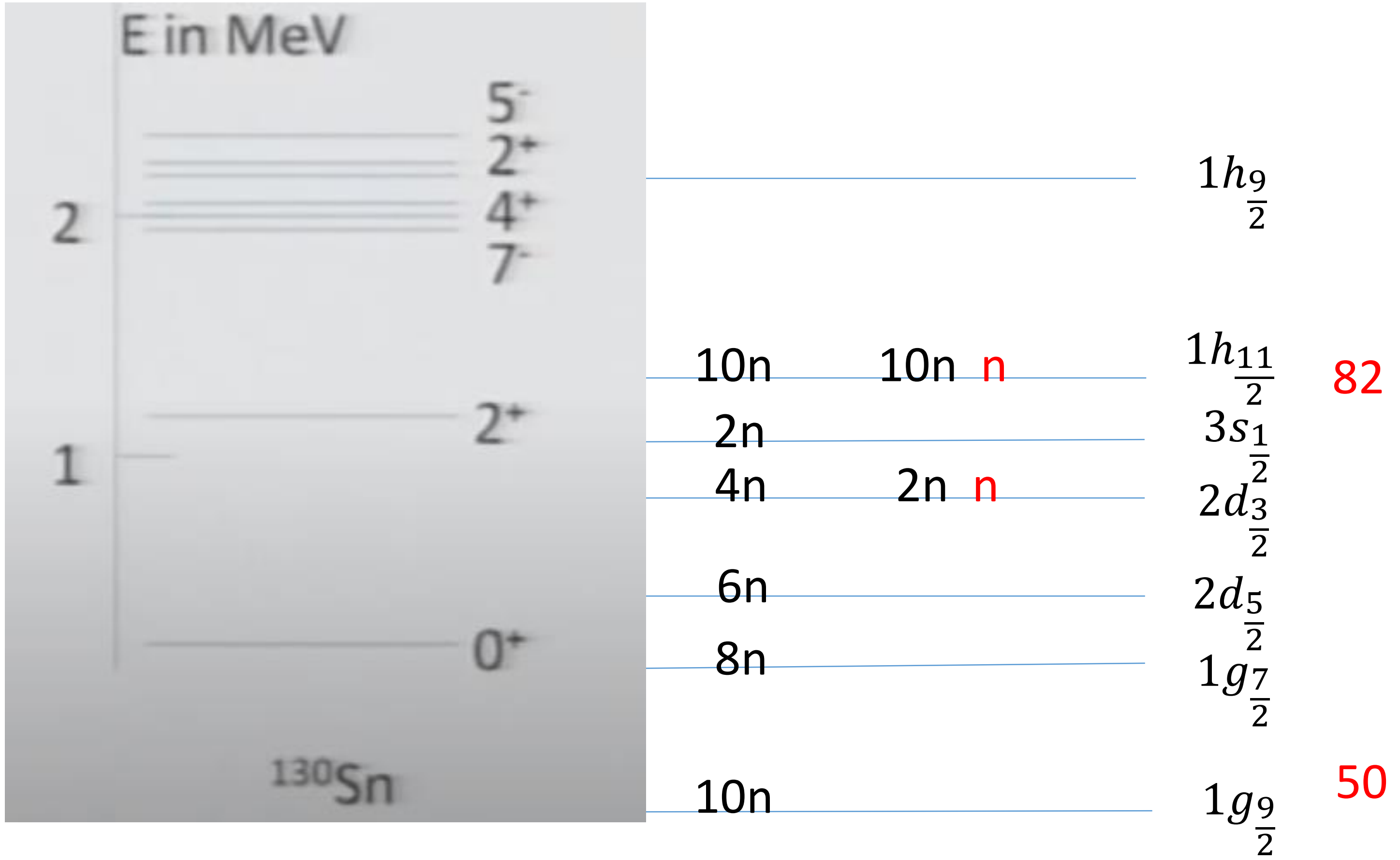


- Energy is pumped into the system to break the pair at $3s_{\frac{1}{2}}$ and to elevate one n to $1h_{\frac{11}{2}}$.
- $j_1 = \frac{11}{2}; j_2 = \frac{1}{2}$ combine to give $5^-, 6^-$. $\Rightarrow$ 5^- state lying above $2\ MeV$ might be explained by this scheme.

Contunued….

Example: $^{130}_{50}Sn(Z = 50, N = 80)$

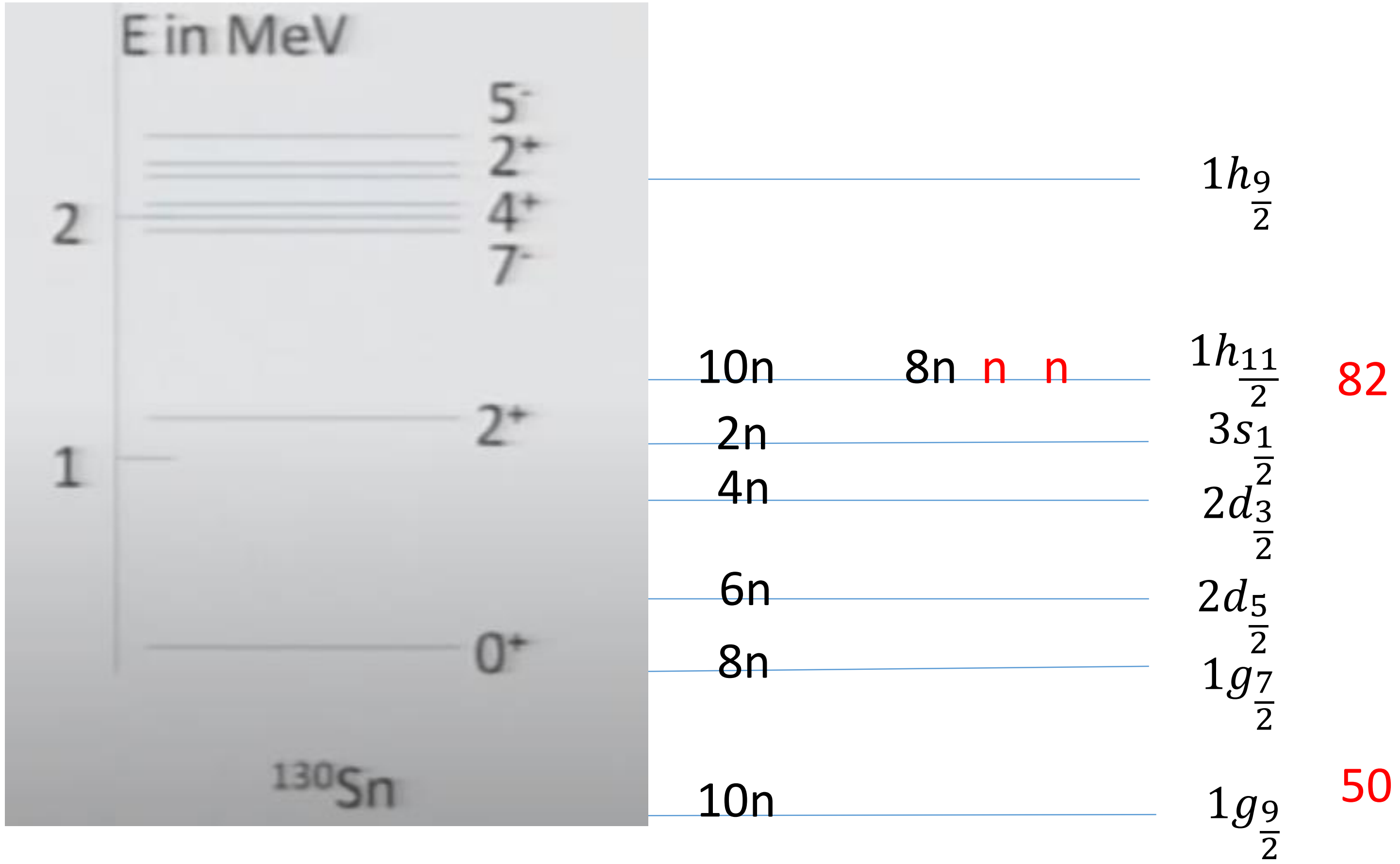
- Energy is pumped into the system to break the pair at $2d_{\frac{3}{2}}$ and to elevate one n to $1h_{\frac{11}{2}}$.
- $j_1 = \frac{11}{2}; j_2 = \frac{3}{2}$ combine to give $7^-, 6^-, 5^-, 4^-$. $\Rightarrow 7^-, 5^-$ states lying Above $2\ MeV$ might be explained by this scheme.



Contunued….

Example: ${}^{130}_{50}Sn(Z = 50, N = 80)$

- Energy is pumped into the system to break the pair at $1h_{\frac{11}{2}}$ and to keep those neutrons in unpaired condition in $1h_{\frac{11}{2}}$.
- $j_1 = \frac{11}{2}; j_2 = \frac{11}{2}$ combine to give $11^+, 10^+, 9^+ \ldots. 1^+, 0^+. \Rightarrow 2^+, 4^+$ states lying above $2\ MeV$ may be explained.



- Remember all the states in the above scheme are not allowed because of identical neutrons.
- It can be shown that only states corresponding to $10^+, 8^+, 6^+, 4^+, 2^+, 0^+$
- Hence $4^+, 2^+$ states lying above $2\ MeV$ can be explained by this scheme.

Identical particles:

$j_1 = 2; j_2 = 2$

$j = 0,1,2,3,4$

$m_{j_1} = -2,-1,0,+1,+2$

$m_{j_2} = -2,-1,0,+1,+2$

$m_j = m_{j_1} + m_{j_2}$

$J^\pi = 0^+, 2^+, 4^+$

Odd values are not allowed as two Fermions are identical sharing the same State.