MODULE-3

- Nuclear Collective Model
- Fermi Gas Model

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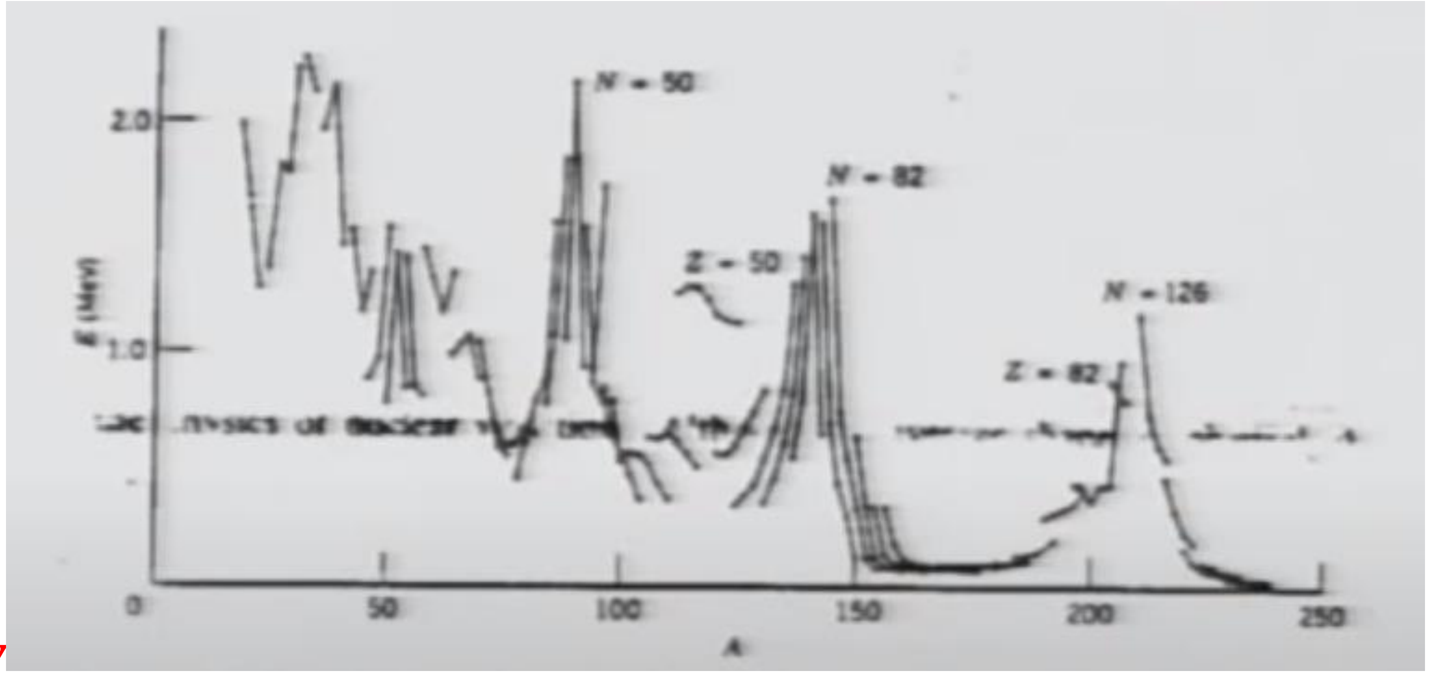
NUCLEAR MODELS(L1)

Collective Model:

- Almost for all e-e nuclei ($A \leq 150, 200 < \mathrm{A} < 220$) it is observed that 1st excited state is 2^+ and it is lying near 1.2 MeV.
- As the number of nucleons increases it may happen so that a large fraction of nucleons would determine the nuclear properties or even the whole nucleus is participating in determining the properties of the nucleus.

Observations:

From the adjacent plot it is seen that $E(2^+)$ value corresponding to nuclei containing magic number of neutrons is much larger compared to the nuclei containing one less or one more number of nucleons.



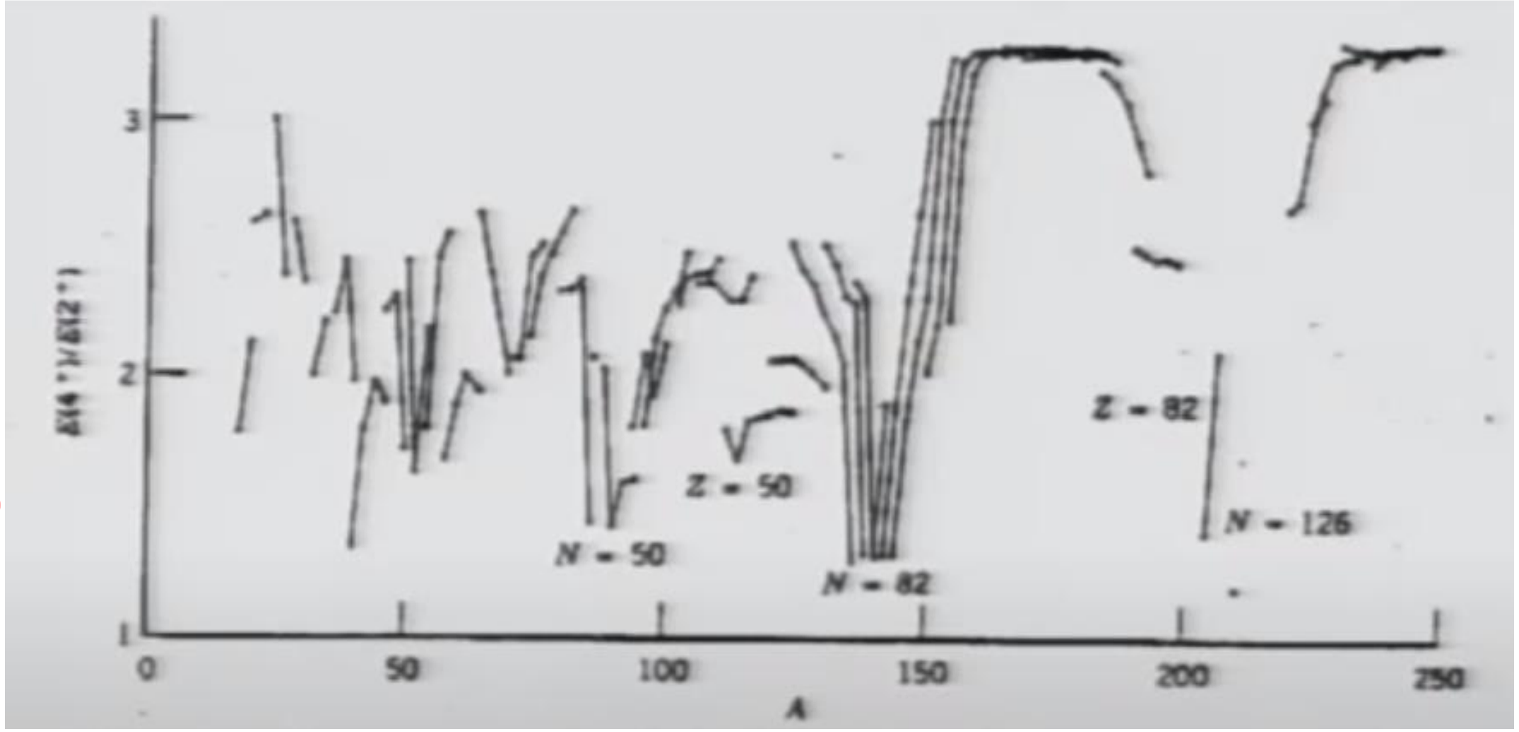
The mean value of $E(2^+)$ is around $1\ MeV$.

It is also observed that $E(2^+) \approx 100\ KeV$ for $150 \leq \mathrm{A} \leq 200$ and $\mathrm{A} \geq 220$.

Collective Model:

Observations:

- From the adjacent plot it is seen that the values of $\frac{E(4^+)}{E(2^+)}$ corresponding to magic numbers are very low.
- For almost all e-e nuclei with $A \leq 150$ it is found that the ratio lies around 2.
- For $150 \leq A \leq 200$ and $A \geq 220$, the ratio is found to be around 3.3.



Conclusions:

- Inter-nucleon interaction in a particular shell can explain some of the excited states of a number of nuclei. For example O, Ca etc.
- For explaining the excited states of e-e nuclei, the pair needs to be broken and for which approximately $2\ MeV$ energy is to be given to the nucleus. To elevate the nucleon to an excited state more energy is needed. Hence 1st excited state must lie above $2\ MeV$ energy.

Collective Model:

Conclusions:

- Observations from last two slides we see that the 1st excited states of almost all e-e nuclei are 2^+ and are found to lie near 1 MeV. Shell model lacks in explaining this feature.
- It may happen so that instead of one or two nucleons more number of nucleons are taking part in determining the properties of the nuclei many more nucleons are participating in this business.
- Actually the whole nucleus may take part in this business.
- Vibrational motion and Rotational motion of the whole nucleus should be taken into consideration. Hence the collective motions of all nucleons considering here give rise to a new model called Collective Model.
- Collective Model is a blending of the features of two models namely – Liquid Drop Model and Shell Model.
- Vibrational & Rotational levels of Nucleus will be discussed in the next class.

Collective Model

Vibrational Motion:

- It is observed from experiments that the 1st excited state lies around 1.2 MeV for most of the e-e nuclei. $(A \leq 150, 200 < \mathrm{A} < 220)$.

Consider the nucleus to be an incompressible liquid drop whose shape can be altered without changing its volume.

When a spherical ball is compressed along its horizontal diameter it bulges along vertical direction and if it is squeezed along vertical direction it bulges along horizontal direction. If this process of compression is continued by an external periodic force, the shape of the nucleus changes periodically resulting in Shape Oscillation/Shape Vibration of the nucleus.

- Since the dimension of the nucleus being microscopic $(\sim fm)$, this vibration may be considered as quantum mechanical phenomenon.
- Hence the energy levels are discrete and quantized.
- Each mode of vibration corresponds to a particular mode of vibration having fixed frequency of vibration.
- Energy levels corresponding to nuclear shape oscillation must lie in between the energy gaps corresponding to Shell Model.

- If the energy separation between two successive energy levels due to shell structure of the nucleus is greater than the energy separation between levels due to shape oscillation of the nucleus for a particular mode of vibration, then the vibrational energy levels will lie in between the energy levels of shell structure.
- Energy of a particular level is, $E = \hbar\omega$, where ω denotes frequency of vibration.

$$R(\theta,\phi) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} a_{\lambda\mu}(t) Y_{\lambda}^{\mu}(\theta,\phi)$$

- $a_{\lambda\mu}(t)$ is time dependent function which takes care of change in shape with time. $Y_{\lambda}^{\mu}(\theta,\phi)$ denotes spherical harmonics.

$$R(\theta,\phi) = R_{avg} + \sum_{\lambda=1}^{\infty} \sum_{\mu=-\lambda}^{\lambda} a_{\lambda\mu}(t) Y_{\lambda}^{\mu}(\theta,\phi)$$

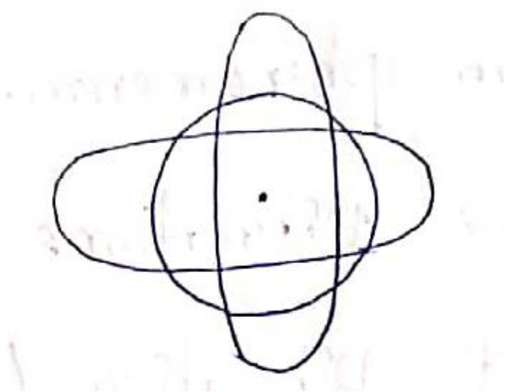
- Here, $Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$; $Y_1^{0,\pm1}(\theta,\phi) = f(\theta,\phi)$

- For $\lambda = 0 \Rightarrow R = R_{avg}$, Hence the shape of the nucleus remains spherical.
- For $\lambda = 1 \Rightarrow$ Dipole Vibration where the CM of the nucleus oscillates keeping the shape of the nucleus spherical. It does not comply with shape oscillation criterion. Angular momentum, $\lambda = 1 \,.\, (E_1 = \hbar\omega_1)$
- For $\lambda = 2 \Rightarrow$ Quadrupole Vibration, Shape changes.

  Angular momentum, $\lambda = 2 \,.\, (E_2 = \hbar\omega_2)$

  Each λ gives different energy levels.

  Parity = $(-1)^2 = +1$

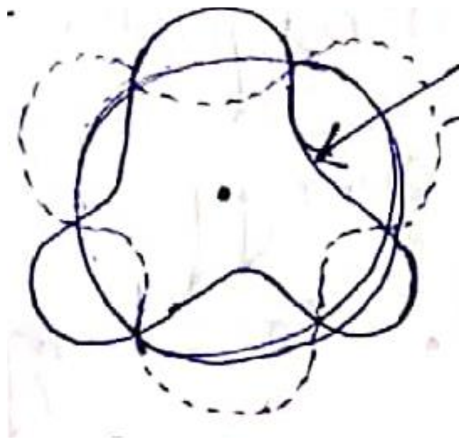


- For $\lambda = 3 \Rightarrow$ Octupole Vibration, Shape changes.

  Angular momentum, $\lambda = 3 \,.\, (E_3 = \hbar\omega_3)$

  Each λ gives different energy levels.

  Parity = $(-1)^3 = -1$



- In all the above cases phonon is the quanta of mechanical vibration.
- Each mode of vibration corresponding to nuclear shape oscillations correspond to a phonon of particular frequency.

$\lambda = 2$

1 phonon

$E_2 = \hbar\omega_2$

Angular momentum, $\lambda = 2^+$

Parity = $(-1)^2 = +1$

2 phonon

$E_2 = 2\hbar\omega_2$

Angular momentum,

$\vec{2} + \vec{2} \rightarrow 0^+, 2^+, 4^+$ (Identical bosons)

3 phonon

$E_2 = 3\hbar\omega_2$

Angular momentum,

$\vec{2} + \vec{2} + \vec{2} \rightarrow 0^+, 2^+, 3^+, 4^+, 6^+$

(Identical bosons)

$\lambda = 3$

1 phonon

$E_2 = \hbar\omega_3$

Angular momentum, $\lambda = 3^-$

Parity = $(-1)^3 = -1$

2 phonon

$E_2 = 2\hbar\omega_3$

Angular momentum,

$\vec{3} + \vec{3} \rightarrow 0^+, 2^+, 4^+, 6^+$

Energy levels of 2 −phonon in $\lambda = 2$ state and

1 −phonon in λ =3 state are close to each other.

$2\hbar\omega_2 = \hbar\,\omega_3 \Rightarrow \omega_2 : \omega_3 = 1:2$

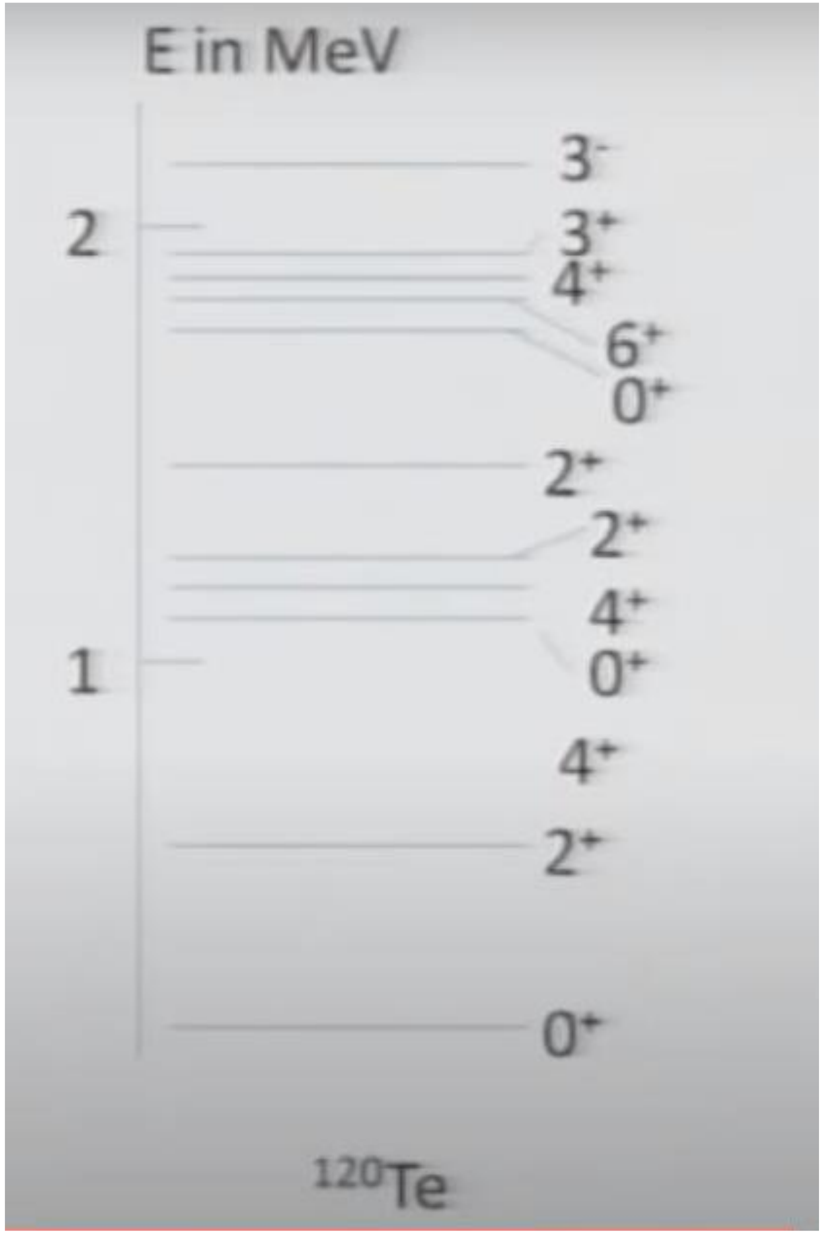
Example: ${}^{120}_{52}Te$

- 1^{st} 2^+ state may be due to 1-phonon in $\lambda = 2$ state
- $0^+, 2^+, 4^+$ states may be due to 2-phonon in $\lambda = 2$ state
- $0^+, 2^+, 3^+, 4^+, 6^+$ states may be due to 3-phonon in $\lambda = 2$ state
- 3^- state may be due to 1-phonon in $\lambda = 3$ state.

  Or

  It might be due to nucleon-nucleon interactions as the energy level is found to be located above 2 MeV.

Shape Oscillation of the nucleus can explain the origin of these Energy levels. The motion of the nucleus as a whole is considered.

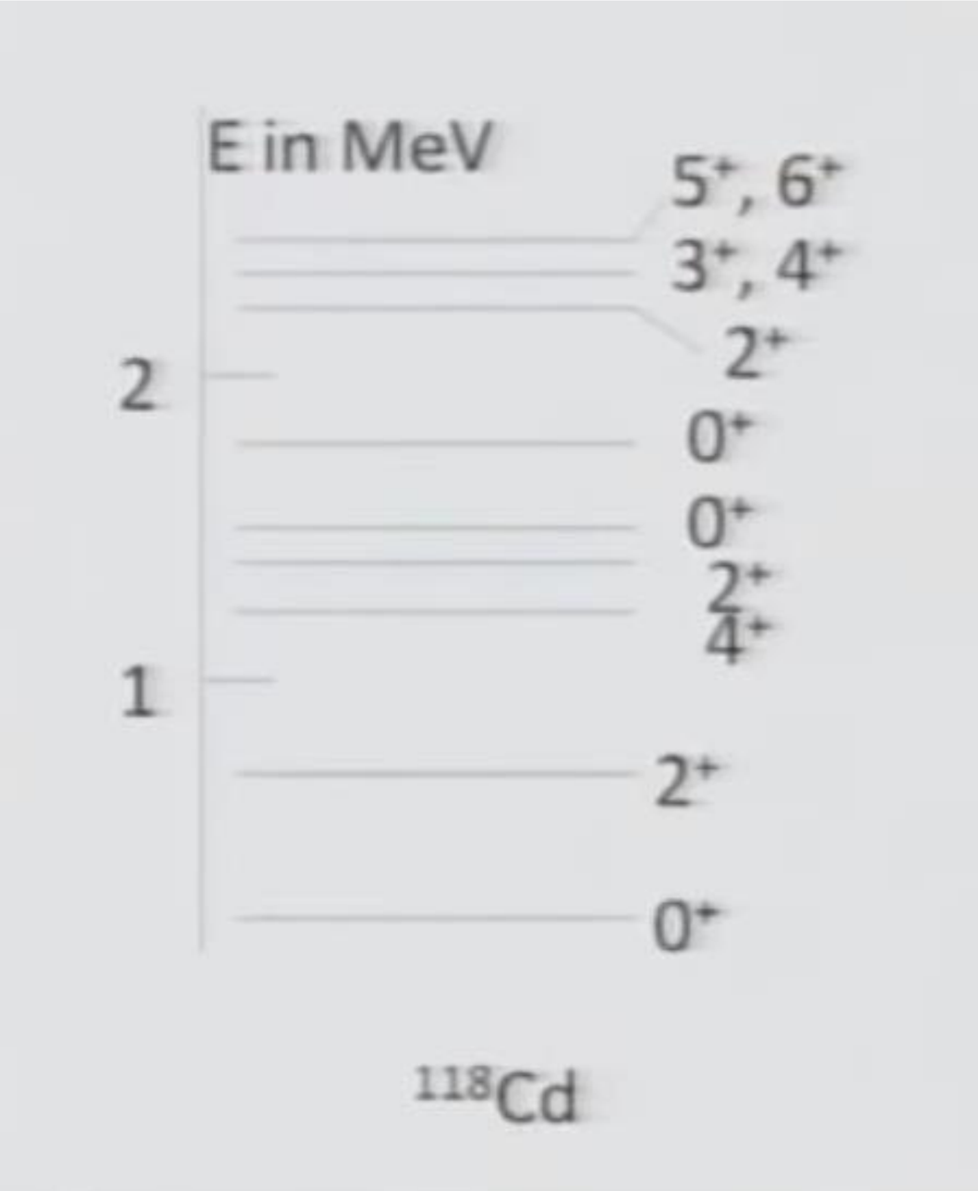


Example: $^{118}_{48}Cd$

Energy levels of $^{118}_{48}Cd$ shown in the adjacent figure can also be explained by Nuclear vibration.(HW)

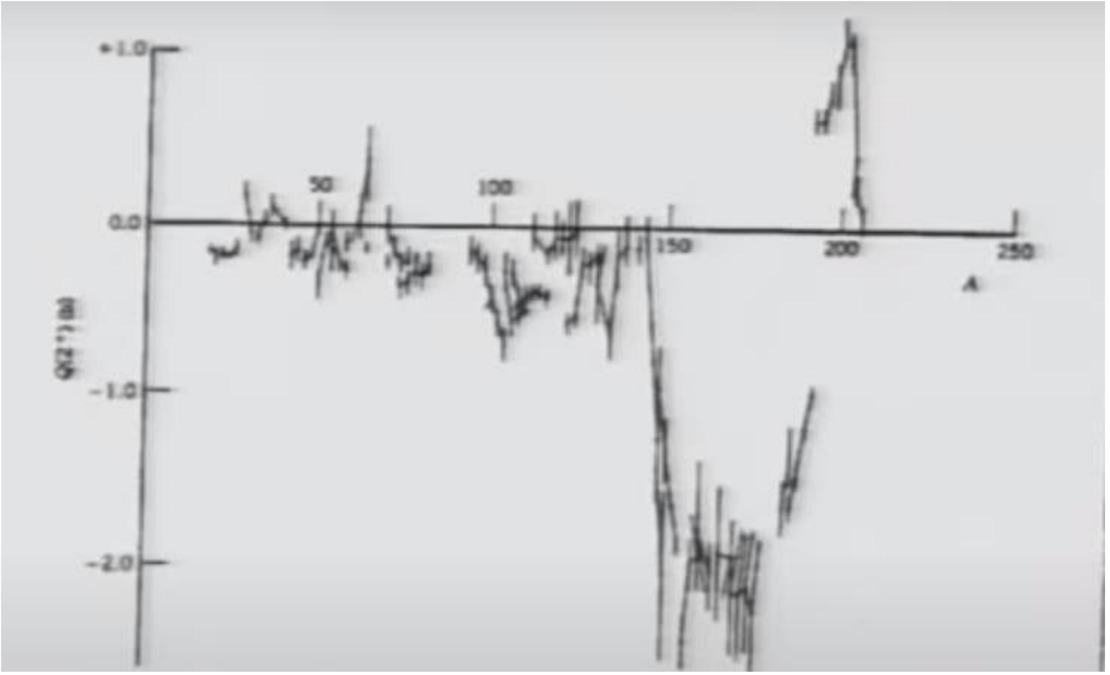
Energy levels can be explained by considering shape oscillation of the given nucleus.

Hence it is concluded that low-lying excited states of e-e nuclei can be explained by considering the vibrational motion of nucleus under Collective Model.



Rotational Motion :

- It is also observed that $E(2^+) \approx 100\ KeV$ for $150 \leq A \leq 200$ and $A \geq 220$.
- There are nuclei lying in the range: $150 \leq A \leq 200$ and $A \geq 220$, it has been found that $\frac{E(4^+)}{E(2^+)}$ lies around 3.3
- The quadrupole moments of e-e nuclei are found to be nearly zero for the whole range of A except $150 \leq A \leq 190$ (-ve) and $190 \leq A \leq 210$ (+ve)



- Large quadrupole moment tells us that equilibrium shape of the nuclei deviates from spherical shape.
- Charge distribution is non-spherical even in the ground state.
- If we rotate a non-spherical nucleus about some appropriate axis, new configurations will be obtained.
- The non-spherical nuclei during rotational motion corresponding to different configuration brings new energy to the system.
- $Q < 0 \Rightarrow Oblate\ spheroid\ \&\ Q > 0 \Rightarrow Prolate\ spheroid$

Example: $^{174}_{74}W$

Rotational KE $E = \frac{L^2}{2I} = \frac{l(l+1)\hbar^2}{2I}$

$l = 0; E = 0$

$l = 1; E = \frac{2\hbar^2}{2I}$ (×)

$l = 2; E = \frac{6\hbar^2}{2I}$

$l = 3; E = \frac{12\hbar^2}{2I}$ (×)

$l = 4; E = \frac{20\hbar^2}{2I}$

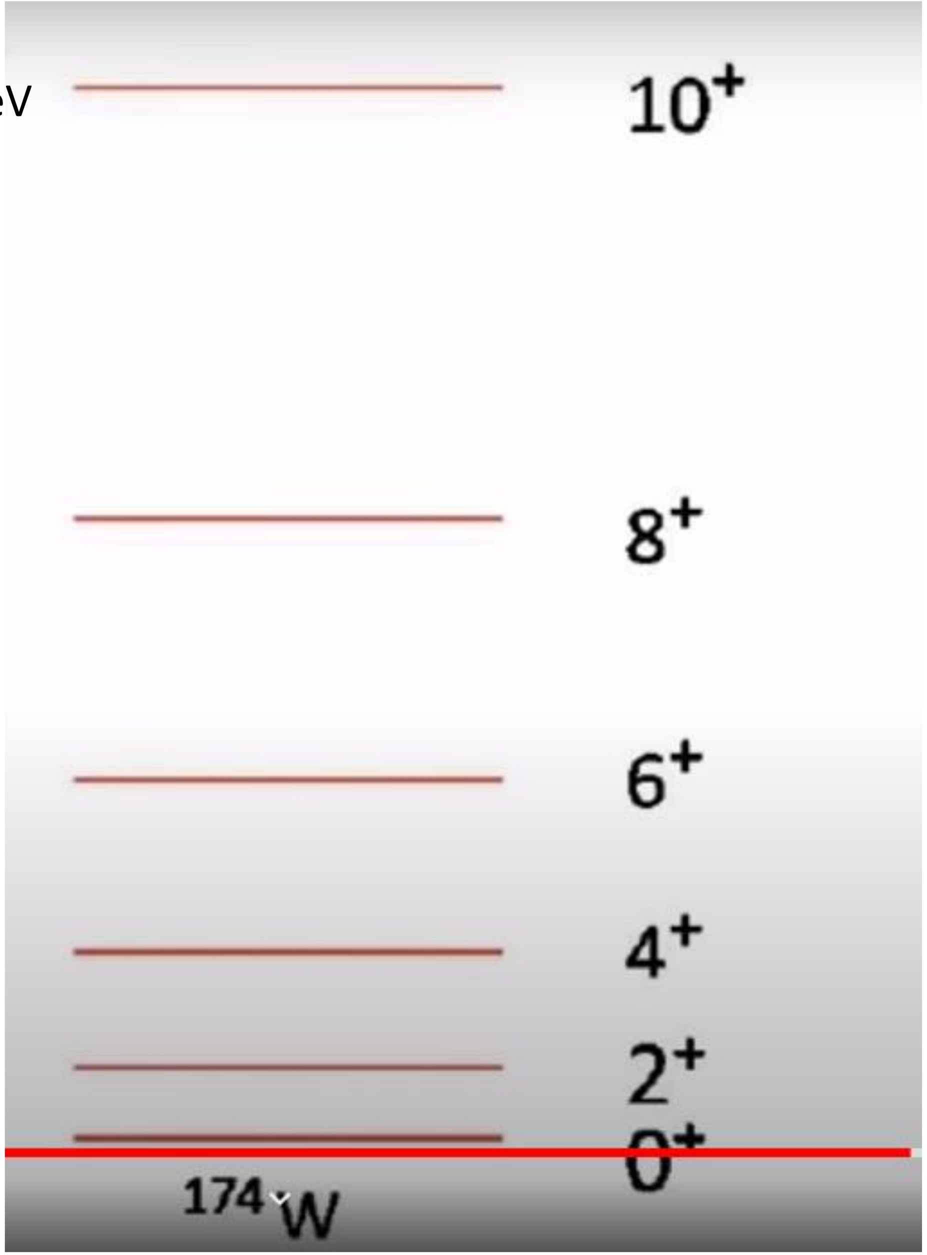
$l = 5; E = \frac{30\hbar^2}{2I}$ (×)

$l = 6; E = \frac{42\hbar^2}{2I}$

- Although the nucleus deviates from spherical shape but its charge distribution still preserves some symmetry called mirror symmetry. This symmetry forbids odd values of angular momentum.

- $\frac{E(4^+)}{E(2^+)} = \frac{20}{6} = 3.33$ which is consistent with the experimental results.
- Energy separation continuously increases as l increases.

E in MeV



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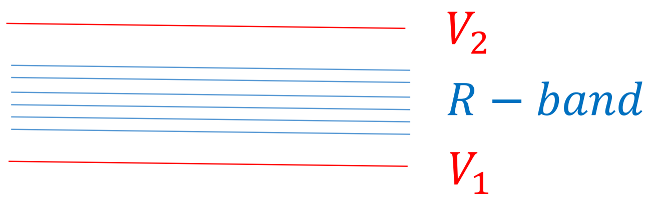


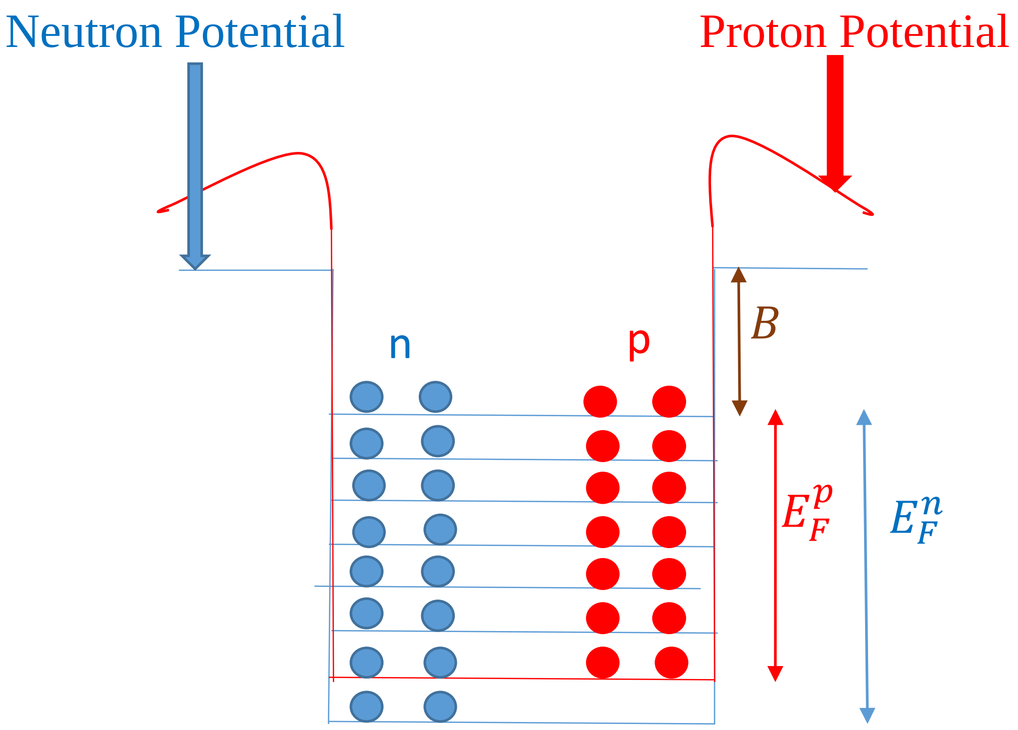
- For $E_2 = 100\ MeV$, we obtain $E_4 = 100 \times 3.3 = 330\ MeV$
- Rotational energy levels are in KeV range.
- Vibrational energy levels are in MeV range.
- Rotational energy levels lie in between vibrational energy levels.
- They are so close to each other that they form a band.
- Taking up the ideas from Shell model and Liquid Drop model together with nuclear vibration and rotations one can develop the idea of Collective Model for explaining the energy levels discussed.
- With these models keeping in mind one can understand many properties of the nuclei, like spin-parity, of ground states, excited states, dipole moments, quadrupole moments and so on in spite of the fact that exact form of Nuclear potential is unknown.

NUCLEAR MODELS(L2)

Fermi Gas Model

Assumptions:

- System of weakly interacting fermions
- Obey Fermi-Dirac statistics
- Obey Pauli's exclusion principle
- Protons and neutrons are moving freely inside the nucleus
- In the 1st approximation nuclear potential is assumed to be constant with sharp edges.
- Neutrons and protons are distinguishable Fermions and they occupy separate potential wells.
- All energy states are filled up by pair of nucleons resulting in no empty states ⇒ No transition between states



- Fermi level is defined as highest occupied energy state denoted by E_F.
- E_F^p and E_F^n denote Fermi levels for protons and neutrons respectively.
- B stands for average BE/nucleon (difference between top of the well and Fermi level) which is about $(7-8)\ MeV$.

Number of nucleon states:

Heisenberg Uncertainty Principle: $\Delta p_x \Delta x \approx \frac{\hbar}{2}$

The volume of one particle phase space $2\pi\hbar$

The number of nucleon states in a volume $V(= \int d^3r)$:

$$\tilde{n} = \frac{\int \int d^3r d^3p}{(2\pi\hbar)^3} = \frac{4\pi V \int_0^{p_m} p^2 dp}{(2\pi\hbar)^3}$$

At temperature, $T = 0$, the nucleus remains in its ground state, the low energy states will be filled up to a maximum momentum called Fermi momentum, $p_F(= p_m)$. Hence, the number of states is:

$$\tilde{n} = \frac{4\pi V \int_0^{p_F} p^2 dp}{(2\pi\hbar)^3} = \frac{4\pi V p_F^3}{(2\pi\hbar)^3 \times 3} \Rightarrow \tilde{n} = \frac{V p_F^3}{6\pi^2\hbar^3}$$

Since an energy state can contain two fermions of the same species, we can have

Neutrons : $N = \frac{V\left(p_F^n\right)^3}{3\pi^2\hbar^3}$ Protons: $Z = \frac{V\left(p_F^p\right)^3}{3\pi^2\hbar^3}$

p_F^n stands for Fermi momentum for neutrons and p_F^p that for protons.

Determination of Fermi momentum:

We know, $R = R_0 A^{\frac{1}{3}}$, Hence, $V = \frac{4}{3}\pi R_0^3 A$

Density of nucleons in a nucleus = number of nucleons in a volume V (taking into account spin of nucleons)

$$n = 2 \times \tilde{n} = 2 \times \frac{V p_F^3}{6\pi^2 \hbar^3} = 2 \times \frac{4}{3}\pi R_0^3 A \times \frac{p_F^3}{6\pi^2\hbar^3} = \frac{4}{9}\frac{R_0^3 A p_F^3}{\pi \hbar^3}$$

Therefore, Fermi momentum p_F:

$$p_F = \left(\frac{9\pi n}{4A}\right)^{\frac{1}{3}} \frac{\hbar}{R_0}$$

Assuming neutron and proton potential wells to be of same radius and $n = N = Z = \frac{A}{2}$, the Fermi momentum is

$$p_F = p_F^n = p_F^p = \left(\frac{9\pi}{8}\right)^{\frac{1}{3}} \frac{\hbar}{R_0} \approx 250\ MeV/c$$

The nucleons move freely with high momentum inside the nucleus.

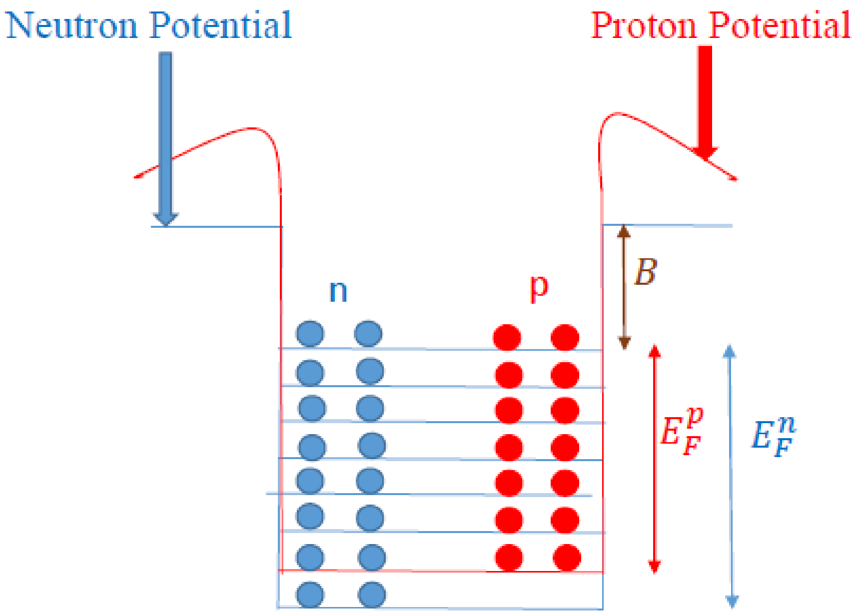
Fermi energy: $E_F = \frac{p_F^2}{2M} \approx 33\ MeV$ where, M = *mass of proton* = $938\ MeV$

- The difference B between the top of the well and the Fermi level is constant for most of the nuclei and is just the average Binding energy, B/A =(7-8) MeV .

  The depth of the potential V_0 and the Fermi energy are independent of the mass number A. Hence,

  $V_0 = E_F + B \approx (33 + 7)MeV = 40MeV$

  The depth of the nuclear potential $V_0 = 40\ MeV$



- For heavy nuclei there is surplus of neutrons. Since, Fermi level for neutrons and protons of a stable nucleus is the same (otherwise $\beta -$ decay would occur), the depth of the potential for neutrons would be greater than that for the protons.$E_F^n > E_F^p$ (see the figure)
- Therefore, protons are less tightly bound to the nucleus compared to neutrons. This happens due to the presence of repulsive Coulomb term in the nuclear potential.

The dependence of the BE on surplus of neutrons may be calculated within the Fermi gas model.

We have to find an expression for average KE per nucleon,

$$\langle E_K \rangle = \frac{\int_0^{E_F} E \frac{dn}{dE} dE}{\int_0^{E_F} \frac{dn}{dE} dE} = \frac{\int_0^{p_F} E \frac{dn}{dp} dp}{\int_0^{p_F} \frac{dn}{dp} dp}$$

Where, $\frac{dn}{dp} = const. \times p^2$, therefore, we obtain,

$$\langle E_K \rangle = \frac{\int_0^{p_F} E\, p^2 dp}{\int_0^{p_F} p^2 dp} = \frac{\int_0^{p_F} \frac{p^2}{2M} p^2 dp}{\int_0^{p_F} p^2 dp} = \frac{3}{5} \frac{p_F^2}{2M} \approx 20\ MeV$$

Total KE of the nucleus, $E_K(N, Z) = N\langle E_K^n \rangle + Z\langle E_K^p \rangle = \frac{3}{10M}\left[N(p_F^n)^2 + Z\left(p_F^p\right)^2\right]$

$$E_K(N, Z) = \frac{3}{10M} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \left[\frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right]$$

Here, we assume the nuclear potential for protons and neutrons to be the same.

The average KE has a minimum at $N = Z$ for fixed mass number A. Remember that N/Z can vary keeping A constant. Hence BE is maximum for $N = Z$.

Now,

$$E_K(N,Z) = \frac{3}{10m}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left[\frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}}\right]$$

Define, $N - Z = \epsilon$ and it is known that $N + Z = A$.

Therefore, $N = \frac{1}{2}(A + \epsilon)$ and $Z = \frac{1}{2}(A - \epsilon)$

Putting these values in the average KE expression we get,

$$E_K(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\left[\frac{(A+\epsilon)^{\frac{5}{3}} + (A-\epsilon)^{\frac{5}{3}}}{2^{\frac{5}{3}}A^{\frac{2}{3}}}\right] = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{A}{2^{\frac{5}{3}}}\left[\left(1+\frac{\epsilon}{A}\right)^{\frac{5}{3}} + \left(1-\frac{\epsilon}{A}\right)^{\frac{5}{3}}\right]$$

$$E_K(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{A}{2^{\frac{5}{3}}}\left[1 + \frac{5}{3}\frac{\epsilon}{A} + \frac{1}{2}\times\frac{5}{3}\times\left(\frac{5}{3}-1\right)\left(\frac{\epsilon}{A}\right)^2 + 1 - \frac{5}{3}\frac{\epsilon}{A} + \frac{1}{2}\times\frac{5}{3}\times\left(\frac{5}{3}-1\right)\left(\frac{\epsilon}{A}\right)^2\right]$$

- Keeping up to 2nd order in ϵ.

Continued….

$$E_K(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{A}{2^{\frac{5}{3}}} \times 2\left[1 + \frac{5}{9}\left(\frac{\epsilon}{A}\right)^2\right]$$

Substituting the value of ϵ $(= N - Z)$ and making some simplifications we obtain,

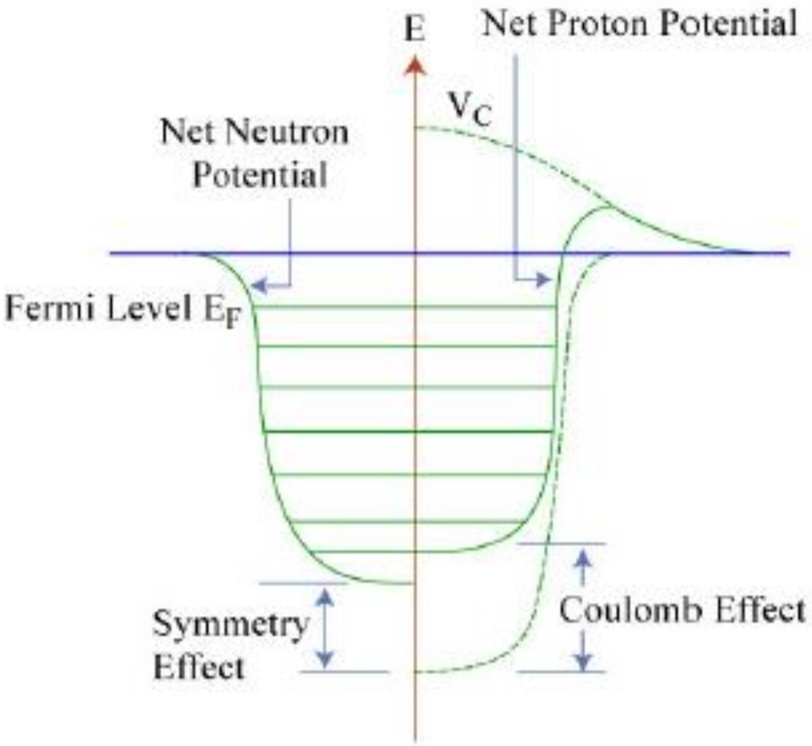
$$E_K(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}\left[A + \frac{5}{9}\frac{(N-Z)^2}{A}\right]$$

- 1st term in the above expression (proportional to A) corresponds to volume energy term of Bethe-Weizsacker mass formula.
- 2nd term (proportional to $\frac{(N-Z)^2}{A}$) corresponds to the asymmetry energy term which grows with surplus of neutrons resulting in decrease in BE of the nucleus.

Problem: Estimate the coefficients of volume energy term and the asymmetry energy term. Comment on your result.

Continued….

- From the adjacent figure it is clear that due to Coulomb potential among protons, the potential well for protons will be different compared to the neutron potential well.
- The proton potential will be a little shallow compared to the neutron well.
- Extra energy levels of neutrons can accommodate more neutrons compared to protons subject to Pauli's exclusion principle.
- Surplus of neutron number over proton number for heavier nuclei can be explained with Fermi gas model.



- This model fails to account for the properties of the low lying nuclear states.
- Small momentum transfers are not permitted as low lying energy states are filled up. But this model is sensitive to large momentum transfer spectrum => properties of excited states can be predicted.